Generalised no-broadcasting for process theories

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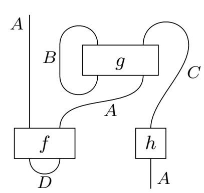
This result appears in the forthcoming textbook [4] as well as in the book chapter [3].

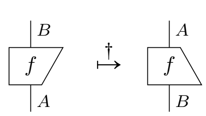
The (universal) *no-broadcasting theorem* states that there exists no quantum process Δ from 1 system to 2, such that, for any quantum state ρ, discarding (i.e. tracing out) either system of $\Delta(\rho)$ yields ρ itself [2]. This theorem has been generalised within the context of generalised probabilistic theories in [1]. Here, we generalise it within the context of *process theories*. For more background on the process theory framework and other details we refer to [3].

1 Background

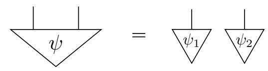
A *string diagram* consists of a collection of boxes which can have some inputs, depicted as wires coming in the bottom of a box, and some outputs, depicted wires coming out of the top. We allow arbitrary connections between boxes, including those from inputs to inputs, and outputs to outputs:



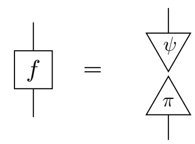
String diagrams can furthermore be reflected vertically:



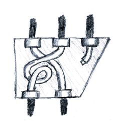
indicating the *adjoint* of the diagram. A *process theory* is an interpretation of all diagrams made up of a fixed collection of boxes (as well as their adjoints) and wires. A two-system state ψ is $\otimes$-*separable* if there exist states ψ_1 and ψ_2 such that:



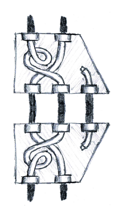
and that a process f is $\circ$-*separable* if there are effect π and state ψ such that:



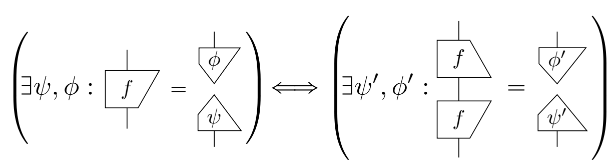
Suppose we have a $\circ$-non-separable process, and imagine that it has some internal structure, say a collection of tubes connecting some inputs to outputs:



If we now compose this process with its vertical reflection, then these internal connections match up:



so one expects the resulting process also to be $\circ$-non-separable. That is:



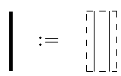
We call this rule *dagger-connectedness*.

Example 1.1. For linear maps, $\circ$-separable means rank-1. So, *dagger-connectedness* follows from the fact that the rank of $f^\dagger \circ f$ is the same as the rank of f. However, this fails for relations. Consider:

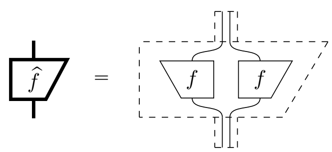
$$R := \{(0,0),(0,1),(1,1)\} \subseteq \{0,1\} \times \{0,1\}$$

Clearly R is $\circ$-non-separable, but $R^\dagger \circ R$ is $\circ$-separable.

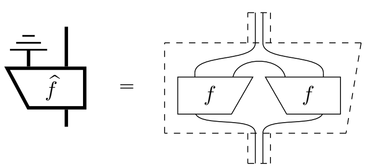
By a *quantum system* we mean a wire of the form:



A *pure quantum process* is of the form:

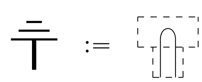


while a general *quantum process* is of the form:



(1)

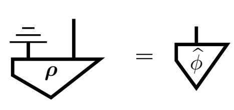
where we used *discarding*:



2 Main result

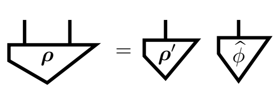
Consider a process theory that admits string diagrams and obeys dagger-connectedness.

Proposition 2.1. If a *reduced state* of $\boldsymbol{\rho}$ (i.e. a state arising from discarding some part of $\boldsymbol{\rho}$) is pure:

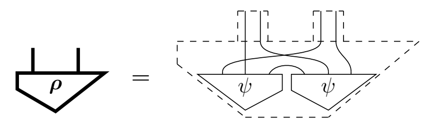


(2)

then $\boldsymbol{\rho}$ $\otimes$-separates as follows:

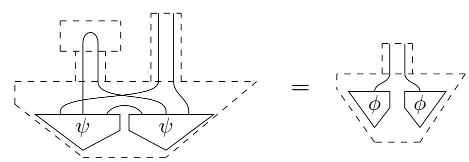


Proof. Writing $\boldsymbol{\rho}$ in the form (1):

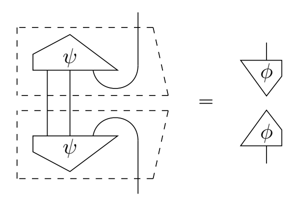


(3)

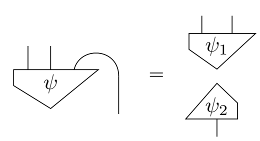
and substituting this into (2) we obtain:



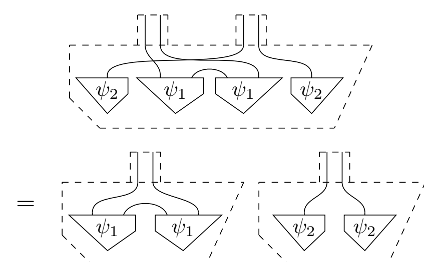
Deforming this equation we get:



Then, by dagger-connectedness there exist ψ_1, ψ_2 s.t:

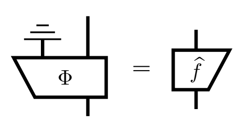


Plugging in to (3) yields the required separation:



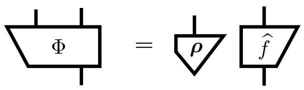
Then, substituting into (2) we can conclude that $\widehat{\psi}_2 = \widehat{\phi}$. □

Proposition 2.2. If a *reduced process* is pure:



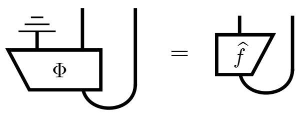
(4)

then it ⊗-separates as follows:

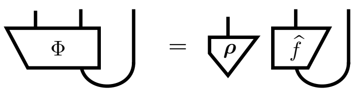


(5)

Proof. Bend the wire in (4):

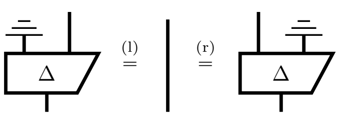


By Proposition 2.1 it separates as follows:



Unbend the wire and we're done. □

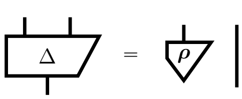
Theorem 2.3. If there is a *quantum broadcasting process*, that is, a quantum process Δ s.t.:



(6)

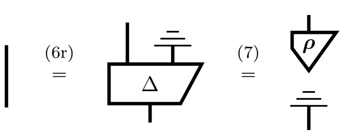
then every plain wire ∘-separates, and consequently, every process ∘-separates. Consequently, non-trivial process theories cannot have such a process.

Proof. By equation (6l) the reduced state of Δ is pure, so by Proposition 2.2 we have:



(7)

for some state $\boldsymbol{\rho}$. Hence it follows that:



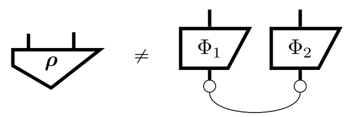
□

3 Discussion

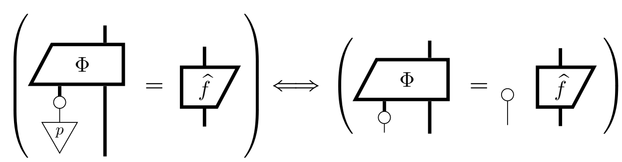
Clearly the crux of this result is the influence of dagger-connectedness on the 'doubled' form of quantum processes.

In fact, we know that dagger-connectedness is strictly stronger than no-broadcasting since the process theory obtained from 'doubling' relations also satisfies no-broadcasting [5], yet relations themselves fail dagger-connectedness, as we saw in Example 1.1.

Furthermore, with the help of so-called 'spiders' for capturing the interaction with classical data, many other results arise from dagger-connectedness [4]. Notable examples are the existence of entangled states:



and the fact that pure process cannot arise from non-trivial convex mixtures of processes:



References

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[2] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher. Noncommuting mixed states cannot be broadcast. *Physical Review Letters*, 76:2818, 1996.

[3] B. Coecke and A. Kissinger. Categorical quantum mechanics I: causal quantum processes. In E. Landry, editor, *Categories for the Working Philosopher*. Oxford University Press, 2016. arXiv:1510.05468.

[4] B. Coecke and A. Kissinger. *Picturing Quantum Processes. A First Course in Quantum Theory and Diagrammatic Reasoning*. Cambridge University Press, 2016.

[5] Chris Heunen and Aleks Kissinger. Can quantum theory be characterized in terms of information-theoretic constraints?, 2016. arXiv:1604.05948.