# Von Neumann Algebras Form a Model for the Quantum Lambda Calculus


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In a recently submitted paper, we present a model for Selinger and Valiron's quantum lambda calculus [5] using von Neumann algebras. The *quantum lambda calculus* is a typed quantum programming language, which contains not only a qubit type, `qbit`, and basic quantum operations (allocation of qubits, measurements and unitary transformations), but also a function type $A \multimap B$ and a 'duplicable' type $!A$ based on linear logic (used to deal with the fact that qubits may not be freely duplicated). *Von Neumann algebras* are the generalisation of matrix algebras (such as $\mathcal{M}_2$, $\mathbb{C}$, and $\mathcal{M}_4 \oplus \mathcal{M}_{37}$) introduced by von Neumann and Murray following their work on giving Quantum Mechanics a solid mathematical basis.

In our model of the quantum lambda calculus, the first-order bits and pieces are interpreted exactly as one might expect: the interpretation of `qbit` is the von Neumann algebra $\mathcal{M}_2$ of $2 \times 2$ complex matrices, the tensor type $\otimes$ is interpreted by the (spatial) tensor product, application of a unitary $U$ on a qubit is modelled by the map $A \mapsto U^*AU,\ \mathcal{M}_2 \to \mathcal{M}_2$ (roughly speaking), and so on — no surprises here.

The interpretation of $A \multimap B$ and $!A$ is less obvious:

$$[\![!A]\!] = \ell^\infty(\mathrm{nsp}([\![A]\!])) \qquad \text{and} \qquad [\![A \multimap B]\!] = (\mathcal{F}\mathcal{J}[\![B]\!])^{*[\![A]\!]}. \tag{1}$$

Here $(-)^{*[\![A]\!]}$ is the free exponential of Kornell [2], that is, the left adjoint to functor $(-) \otimes [\![A]\!]$ on the category $\mathbf{vNA}_{\mathrm{MIU}}$ of von Neumann algebras and normal unital $*$-homomorphisms (the 'structure preserving' maps). This makes $(\mathbf{vNA}^{\mathrm{op}}_{\mathrm{MIU}}, \otimes, \mathbb{C})$ into a monoidal closed category. The other symbols from (1) come from the monoidal adjunctions shown below.

$$(\mathbf{Set}, \times, 1) \underset{\mathrm{nsp}}{\overset{\ell^\infty}{\rightleftarrows}} (\mathbf{vNA}^{\mathrm{op}}_{\mathrm{MIU}}, \otimes, \mathbb{C}) \underset{\mathcal{F}}{\overset{\mathcal{J}}{\rightleftarrows}} (\mathbf{vNA}^{\mathrm{op}}_{\mathrm{CPsU}}, \otimes, \mathbb{C}) \tag{2}$$

Here $\mathbf{vNA}_{\mathrm{CPsU}}$ is the category of von Neumann algebras with normal completely positive subunital maps (representing quantum processes), which has $\mathbf{vNA}_{\mathrm{MIU}}$ as subcategory, $\mathcal{J}\colon \mathbf{vNA}^{\mathrm{op}}_{\mathrm{MIU}} \to \mathbf{vNA}^{\mathrm{op}}_{\mathrm{CPsU}}$ is simply the inclusion functor, and $\ell^\infty$ is the functor which assigns to a set $X$ the von Neumann algebra of bounded functions from $X$ to $\mathbb{C}$.

The functor nsp is the right adjoint to $\ell^\infty$, and can be concretely described (it assigns to a von Neumann algebra the set of normal multiplicative states). The functor $\mathcal{F}$ is the right adjoint to $\mathcal{J}$ and is known to exist by the Adjoint Functor Theorem, but admits no better description, as far as we know.

While $[\![A \multimap B]\!]$ defies concrete description, the interpretation of $!(A \multimap B)$ could not have been crisper:

$$[\![!(A \multimap B)]\!] = \ell^\infty(\{ \text{ normal CPsU-maps } f\colon [\![B]\!] \longrightarrow [\![A]\!] \})$$


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We should probably mention that the first adjunction in (2) is a linear-non-linear model for intuitionistic linear logic, and that, because $\mathbf{vNA}^{\mathrm{op}}_{\mathrm{CPsU}}$ is isomorphic to the Kleisli category of the monad $\mathcal{F} \circ \mathcal{J}$ on $\mathbf{vNA}^{\mathrm{op}}_{\mathrm{MIU}}$, the second adjunction is reminiscent of Moggi's model for a lambda calculus with side-effects. Together, these adjunctions constitute what Selinger and Valiron call a *concrete model of the quantum lambda calculus* in [6, §1.6.8].

Our model is not the first model of the quantum lambda calculus; this honour goes to Malherbe who gave one using presheaves in his thesis [3]. To our knowledge, only two other denotational models have been given: one by Hasuo and Hoshino [1] via Geometry of Interaction, and one by Pagani et al. [4] based on methods from quantitative semantics.

A major part of our paper is devoted to the proof that our model is adequate with respect to the operational semantics. Technically, this makes our model the first known adequate model for Selinger and Valiron's quantum lambda calculus. After all, Malherbe concentrated on the existence of a model, so it is not known whether his model is adequate, and while both models in [1] and in [4] are adequate, they are models for variants of Selinger and Valiron's language.

We should note that while it is possible to extend the quantum lambda calculus with recursion, we have not yet been able to include it in our model, unlike [1, 4].

We believe that our interpretation of the quantum lambda calculus using von Neumann algebras is clean and direct, and we suspect that von Neumann algebras might turn out to be the most appropriate structures to interpret quantum programs between possibly infinite dimensional (and higher-order) data types, but this is up to history.

## References


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