An operational resource theory of purity

Giulio Chiribella
giulio@cs.hku.hk
Department of Computer Science, University of Hong Kong, Hong Kong

Carlo Maria Scandolo
carlomaria.scandolo@cs.ox.ac.uk
Department of Computer Science, University of Oxford, Oxford, UK

A successful approach to the foundations of thermodynamics is to consider purity as a resource. Operationally, this can be done in different ways, depending on which set of operations is regarded as "free", or easy to implement. In classical and quantum theory, all the reasonable choices of free operations lead to the same ordering of states, characterised by the majorisation criterion. But what are the roots of such an equivalence? In this paper we address the question in the framework of general probabilistic theories. For arbitrary theories a notion of purity as a resource can be defined by choosing random reversible channels as free operations. For theories satisfying the axioms of Causality, Purity Preservation, Purification, Pure Sharpness, and one additional axiom, Permutability/Strong Symmetry, we show that one can put forward two alternative notions of purity as a resource: one where free operations are unital channels, and another where free operations are generated by reversible interactions with an environment in the invariant state. These axioms guarantee that all the above resource theories are equivalent, i.e. they all lead to the same (pre)ordering relations between states. For theories satisfying the five axioms we show that the notion of purity as a resource is completely characterised by a majorisation criterion, in the very same way as it is in quantum theory.

1 Introduction

Thermodynamics is one of the most successful paradigms of physics, with applications ranging from engineering to chemistry, up to computation and biology. In recent years, developments in the field of nanotechnology have raised novel questions about what thermodynamic transformations are possible far from the thermodynamic limit [20]. A promising way to address this new regime is to adopt the approach of resource theories [28, 6]. Such an approach is not limited to quantum theory, but instead it is a structural framework for capturing the notion of resource on operational grounds [18, 19]. The idea is to regard a set of operation as "free", and the ability to convert a state into another by means of free operations as a criterion for resourcefulness. Given that most systems approach thermal equilibrium spontaneously, it is natural to define equilibrium states as free states. For example, at low temperatures it is natural to regard the microcanonical ensemble as the free state—for a quantum system with degenerate Hamiltonian, this means that the equilibrium state is the maximally mixed state $\chi = \frac{1}{d}I$, where d is the dimension of the system, and I is the identity matrix. Now, to define a resource theory, one has to specify a set of free operations. But which operations? There are at least three natural choices:

1. the so-called *noisy operations*, generated by preparing the microcanonical state $\chi = \frac{1}{d}I$, performing unitary operations, and discarding systems [27];
2. all the quantum operations that preserve the microcanonical state χ. These are a more general class of channels, called *unital channels* [30, 33];

Submitted to:
QPL 2016

3. *random unitary (RU) channels* [38, 39, 40] $\mathscr{R}(\rho)=\sum_j p_j U_j \rho U_j^\dagger$, where $\{p_j\}$ is a probability distribution, and U_j is a unitary operator for every j.

Now, the important fact is that these three different choices of free operations induce the same notion of resource—mathematically, the same preorder on the set of quantum states [21]. But is the equivalence specific to quantum theory? To address this question, we explore different notions of purity in the realm of general probabilistic theories (GPTs) [24, 5, 3, 8] (see also the contributed volume [14]). More specifically, we consider theories that share some features with quantum theory, corresponding to the axioms of Causality, Purity Preservation, Purification, and Pure Sharpness, previously adopted in our work [13]. These axioms guarantee that every state can be diagonalised, i.e. written as a convex combination of perfectly distinguishable pure states. Then we add one last axiom, Strong Symmetry [4], stating that for every two maximal sets of perfectly distinguishable pure states there exists a reversible channel connecting them. In such a setting we consider the resource theory where free operations are *random reversible (RaRe) channels* [12], and we study how majorisation characterises the preorder induced by these free operations. In Ref. [13] we showed that Strong Symmetry is sufficient for the equivalence between the majorisation preorder and the preorder defined by RaRe channels. Here we prove that Strong Symmetry is also *necessary* for the validity of the majorisation criterion. Moreover, we show that, in the context of our axioms, Strong Symmetry is fully equivalent to Hardy's Permutability axiom [25, 26], which stipulates that every permutation of a maximal set of perfectly distinguishable pure states can be implemented by a reversible channel.

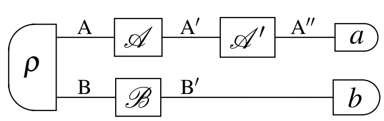
Finally, we analyse the other definitions of purity that can be given in GPTs and their relationship to majorisation. More specifically, we give operational definitions of *noisy operations* and *unital channels*. If Permutability/Strong Symmetry holds, all the three possible choices of free operations lead to the same preorder on states. As a by-product, we establish inclusions between the three sets of free operations: a RaRe channel is a noisy operation, and a noisy operation is a unital channel.

The paper is structured as follows. In section 2 we give a brief review of the framework to study GPTs, and in section 3 we introduce the axioms and list their main consequences. In section 4 we give three different definitions of resource theories of purity, and we establish some inclusions between their sets of free operations. Finally in section 5 we study the role of majorisation in characterising the three resource theories of purity. Conclusions are drawn in section 6.

2 Framework

We work in the variant of GPTs known as operational-probabilistic theories (OPTs) [8, 9, 25, 26, 7, 10], which makes use of a graphical language borrowed from symmetric monoidal categories [1, 15, 16, 36, 17]. We will list here only the main features and notions of this formalism.

Physical processes can be combined in sequence or in parallel to build circuits, such as



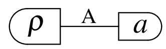
.

Here, A, A′, etc. are *systems*, ρ is a bipartite *state*, $\mathscr{A}$, $\mathscr{A}'$ and $\mathscr{B}$ are *transformations*, a and b are *effects*. We denote by

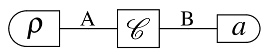
- $\mathsf{St}(\mathrm{A})$ the set of states of system A;
- $\mathsf{Eff}(\mathrm{A})$ the set of effects on A;

- $\mathsf{Transf}(\mathrm{A},\mathrm{B})$ the set of transformations from A to B, and by $\mathsf{Transf}(\mathrm{A})$ the set of transformations from A to A;
- $\mathrm{A}\otimes\mathrm{B}$ the composition of systems A and B;
- $\mathscr{A}\otimes\mathscr{B}$ the parallel composition (or tensor product) of the transformations $\mathscr{A}$ and $\mathscr{B}$.

Every physical theory admits a particular system, the trivial system I (mathematically, the unit of the tensor product), corresponding to the degrees of freedom ignored by the theory. States (resp. effects) are transformations with the trivial system as input (resp. output). Circuits with no external wires, like the one above, represent probabilities. We will often make use of the short-hand notation $(a|\rho)$ to denote

$(a|\rho)\ :=$  ,

and of the notation $(a|\mathscr{C}|\rho)$ to mean

$(a|\mathscr{C}|\rho)\ :=$  .

We identify the scalar $(a|\rho)$ with a real number in the interval $[0,1]$, representing the probability of a joint occurrence of the state ρ and the effect a.

The fact that scalars are real numbers induces a notion of sum for transformations, so that the sets $\mathsf{St}(\mathrm{A})$, $\mathsf{Transf}(\mathrm{A},\mathrm{B})$, and $\mathsf{Eff}(\mathrm{A})$ become spanning sets of real vector spaces. In this paper we will restrict our attention to finite systems, i.e. systems for which the spanned vector spaces are finite-dimensional. We will use the topology induced by probabilities, by which one has $\lim_{n\to+\infty}\mathscr{C}_n=\mathscr{C}$, where $\mathscr{C}_n,\mathscr{C}\in\mathsf{Transf}(\mathrm{A},\mathrm{B})$, if and only if

$$\lim_{n\to+\infty}(E|\mathscr{C}_n\otimes\mathscr{I}_{\mathrm{R}}|\rho)=(E|\mathscr{C}\otimes\mathscr{I}_{\mathrm{R}}|\rho)\quad\forall\mathrm{R},\forall\rho\in\mathsf{St}(\mathrm{A}\otimes\mathrm{R}),\forall E\in\mathsf{Eff}(\mathrm{B}\otimes\mathrm{R}).$$

A *test* from A to B is a collection of transformations $\{\mathscr{C}_i\}_{i\in\mathsf{X}}$ from A to B, which can occur in an experiment with outcomes in X. If A (resp. B) is the trivial system, the test is called a *preparation-test* (resp. *observation-test*). If X contains a single outcome, we say that the test is *deterministic*. We will refer to deterministic transformations as *channels*. A channel $\mathscr{U}$ from A to B is called *reversible* if there exists a channel $\mathscr{U}^{-1}$ from B to A such that $\mathscr{U}^{-1}\mathscr{U}=\mathscr{I}_{\mathrm{A}}$ and $\mathscr{U}\mathscr{U}^{-1}=\mathscr{I}_{\mathrm{B}}$, where $\mathscr{I}_{\mathrm{S}}$ is the identity channel on a system S. If there exists a reversible channel transforming A into B, we say that A and B are *operationally equivalent*, denoted by $\mathrm{A}\simeq\mathrm{B}$. The composition of systems is required to be *symmetric*, meaning that $\mathrm{A}\otimes\mathrm{B}\simeq\mathrm{B}\otimes\mathrm{A}$.

A state $\chi\in\mathsf{St}(\mathrm{A})$ is called *invariant* if $\mathscr{U}\chi=\chi$, for every reversible channel $\mathscr{U}$. In general, invariant states may not exist.

We can define pure transformations based on the notion of *coarse-graining*, i.e. the operation of joining two or more outcomes of a test into a single outcome. More precisely, a test $\{\mathscr{C}_i\}_{i\in\mathsf{X}}$ is a *coarse-graining* of the test $\{\mathscr{D}_j\}_{j\in\mathsf{Y}}$ if there is a partition $\{\mathsf{Y}_i\}_{i\in\mathsf{X}}$ of Y such that $\mathscr{C}_i=\sum_{j\in\mathsf{Y}_i}\mathscr{D}_j$ for every $i\in\mathsf{X}$. In this case, we say that $\{\mathscr{D}_j\}_{j\in\mathsf{Y}}$ is a *refinement* of $\{\mathscr{C}_i\}_{i\in\mathsf{X}}$. The refinement of a given transformation is defined via the refinement of a test: if $\{\mathscr{D}_j\}_{j\in\mathsf{Y}}$ is a refinement of $\{\mathscr{C}_i\}_{i\in\mathsf{X}}$, then the transformations $\{\mathscr{D}_j\}_{j\in\mathsf{Y}_i}$ are a refinement of the transformation $\mathscr{C}_i$. A transformation $\mathscr{C}\in\mathsf{Transf}(\mathrm{A},\mathrm{B})$ is *pure* if it has only trivial refinements, namely refinements $\{\mathscr{D}_j\}$ of the form $\mathscr{D}_j=p_j\mathscr{C}$, where $\{p_j\}$ is a probability distribution. Pure transformations are those for which the experimenter has maximal information about the evolution of the system. We denote the set of pure states (resp. effects) of system A as $\mathsf{PurSt}(\mathrm{A})$ (resp. $\mathsf{PurEff}(\mathrm{A})$). As usual, non-pure states are called *mixed*.

The pairing between states and effects leads naturally to a notion of norm. We define the norm of a state ρ as $\|\rho\| := \sup_{a\in\mathsf{Eff}(\mathrm{A})}(a|\rho)$. Similarly, the norm of an effect a is defined as $\|a\| := \sup_{\rho\in\mathsf{St}(\mathrm{A})}(a|\rho)$. We will use a subscript 1 to denote the set of normalised (i.e. with unit norm) states and effects. For instance the set of normalised states of A will be denoted by $\mathsf{St}_1(\mathrm{A})$, and so on.

Definition 1. Let $\rho\in\mathsf{St}_1(\mathrm{A})$. A normalised state σ is *contained* in ρ if we can write $\rho = p\sigma + (1-p)\tau$, where $p\in(0,1]$ and τ is another normalised state.

Definition 2. We say that two transformations $\mathscr{A},\mathscr{A}'\in\mathsf{Transf}(\mathrm{A},\mathrm{B})$ are *equal upon input* of the state $\rho\in\mathsf{St}_1(\mathrm{A})$ if $\mathscr{A}\sigma = \mathscr{A}'\sigma$ for every state σ contained in ρ. In this case we will write $\mathscr{A} =_\rho \mathscr{A}'$.

3 Axioms and their consequences

Here we provide an overview of the axioms adopted in this paper.

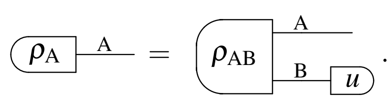
Axiom 1 (Causality [8, 9])**.** *For every preparation-test* $\{\rho_i\}_{i\in\mathsf{X}}$, *and for all observation-tests* $\{a_j\}_{j\in\mathsf{Y}}$ *and* $\{b_k\}_{k\in\mathsf{Z}}$ *we have*

$$\sum_{j\in\mathsf{Y}}(a_j|\rho_i) = \sum_{k\in\mathsf{Z}}(b_k|\rho_i),$$

for all $i\in\mathsf{X}$.

Causality is equivalent to the requirement that, for every system A, there exists a unique deterministic effect u_A on A (or simply u, when no ambiguity can arise) [8]. This implies that all observation-tests $\{a\}_{i\in\mathsf{X}}$ are normalised, namely $\sum_{i\in\mathsf{X}} a_i = u$.

Thanks to the uniqueness of u, it is possible to define the *marginal state* of a bipartite state ρ_{AB} on system A as



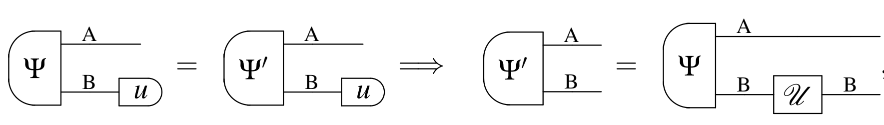
In this case we will also write $\rho_\mathrm{A} := \mathrm{Tr}_\mathrm{B}\rho_{\mathrm{AB}}$, calling u_B as Tr_B; we will tend to keep the notation Tr in formulas where the deterministic effect is directly applied to a state, e.g. $\mathrm{Tr}\,\rho := (u|\rho)$.

In a causal theory the norm of a state ρ takes the simpler expression $\|\rho\| = \mathrm{Tr}\,\rho$, and all states are proportional to normalised states [8]. A transformation $\mathscr{A}\in\mathsf{Transf}(\mathrm{A},\mathrm{B})$ is a channel if and only if it preserves the deterministic effect, that is $u_\mathrm{B}\mathscr{A} = u_\mathrm{A}$.

Another consequence is that all the sets $\mathsf{St}(\mathrm{A})$, $\mathsf{Transf}(\mathrm{A},\mathrm{B})$, and $\mathsf{Eff}(\mathrm{A})$ are *convex*.

The second axiom is Purification, which characterises all physical theories admitting a level of description where all deterministic processes are pure and reversible. A *pure* state $\Psi\in\mathsf{PurSt}_1(\mathrm{A}\otimes\mathrm{B})$ is a *purification* of a state $\rho\in\mathsf{St}_1(\mathrm{A})$ with purifying system B if ρ is the marginal on A of Ψ.

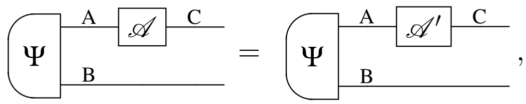
Axiom 2 (Purification [8, 9])**.** *Every state has a purification, and two purifications of the same state with the same purifying system differ by a reversible channel on the purifying system.*



where $\mathscr{U}$ *is a reversible channel.*

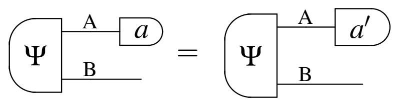
Purification enables us to link equality upon input (as in definition 2) to equality on purifications (cf. theorem 7 of Ref. [8]).

Proposition 1. *Let ρ be a state of system* A *and let $\Psi \in \mathsf{St}_1(\mathrm{A} \otimes \mathrm{B})$ be a purification of ρ. Then, for every pair of transformations $\mathscr{A}$ and $\mathscr{A}'$, from* A *to* C*, if*



then $\mathscr{A} =_\rho \mathscr{A}'$.

If system C *is trivial, then one has the full equivalence: for every pair of effects a and a'*



if and only if $a =_\rho a'$.

We introduce the axiom of Purity Preservation, stating that information cannot be lost when composing transformations on which we have maximal information.

Axiom 3 (Purity Preservation [11])**.** *Sequential and parallel compositions of pure transformations yield pure transformations.*

The final axiom is Pure Sharpness, which guarantees that every system possesses at least one elementary property, in the sense of Piron [35].

Axiom 4 (Pure Sharpness [13])**.** *For every system* A *there exists at least one pure effect occurring with unit probability on some state.*

Combining the four axioms presented so far, one can obtain important strong structural results. The first result is a duality between normalised pure states and normalised pure effects (see propositions 8 and 10 of Ref. [13]).

Proposition 2. *There is a bijective correspondence between normalised pure states and normalised pure effects. Specifically, if $\alpha \in \mathsf{PurSt}_1(\mathrm{A})$, there exists a unique $\alpha^\dagger \in \mathsf{PurEff}_1(\mathrm{A})$ such that $\left(\alpha^\dagger|\alpha\right) = 1$.*

The four axioms also guarantee that every state of a non-trivial system can be diagonalised, i.e. written as a convex combination of perfectly distinguishable pure states[1] [13]. The probabilities arising in the diagonalisation will be called the *eigenvalues* of the state.

We conclude our overview with two additional axioms that will be important later in the paper:

Axiom 5 (Permutability [25, 26])**.** *Every permutation of a maximal*[2] *set of perfectly distinguishable* pure *states can be implemented by a reversible channel.*

Axiom 6 (Strong Symmetry [4])**.** *The group of reversible channels acts transitively on maximal sets of perfectly distinguishable* pure *states.*

Clearly Strong Symmetry implies Permutability. Quite remarkably, the converse is also true, provided that the four axioms discussed before are satisfied:

Proposition 3. *In a theory satisfying Causality, Purity Preservation, Purification, and Pure Sharpness, the following are equivalent.*

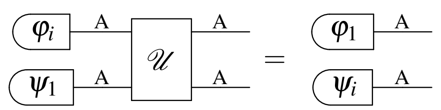
1. *The theory satisfies Strong Symmetry.*

[1] The states $\{\rho_i\}_{i=1}^n$ are *perfectly distinguishable* if there exists an observation-test $\{a_i\}_{i=1}^n$ such that $\left(a_i|\rho_j\right) = \delta_{ij}$.

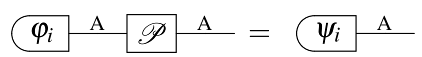
[2] A set of perfectly distinguishable states $\{\rho_i\}_{i=1}^n$ is *maximal* if there is no other state ρ_{n+1} such that the states $\{\rho_i\}_{i=1}^{n+1}$ are perfectly distinguishable.

2. *The theory satisfies Permutability.*

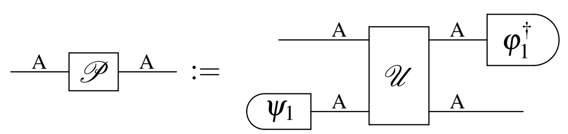
Proof. Let us prove that Permutability implies Strong Symmetry. The first part of the proof is similar to the one of theorem 30 of Ref. [25]. Consider two maximal sets of perfectly distinguishable pure states $\{\varphi_i\}_{i=1}^n$ and $\{\psi_j\}_{j=1}^m$. Assuming Permutability, we will show that $m = n$ and that there exists a reversible channel $\mathscr{U}$ such that $\varphi_i = \mathscr{U}\psi_i$, for all $i = 1,\ldots,n$. First of all, note that the states $\{\varphi_i \otimes \psi_j\}$ are pure (by Purity Preservation) and perfectly distinguishable. Indeed, if $\{a_i\}_{i=1}^n$ and $\left\{a'_j\right\}_{j=1}^m$ are the perfectly distinguishing tests for $\{\varphi_i\}_{i=1}^n$ and $\{\psi_j\}_{j=1}^m$ respectively, the effects $\left\{a_i \otimes a'_j\right\}$ make up a perfectly distinguishing test for $\{\varphi_i \otimes \psi_j\}$. This is because $\sum_{i,j} a_i \otimes a'_j = u \otimes u$, and this is a sufficient condition for a set of effects to be an observation-test, thanks to Purification [9]. We can extend $\{\varphi_i \otimes \psi_j\}$ to a maximal set of perfectly distinguishable pure states for the composite system $\mathrm{A}\otimes\mathrm{A}$. Then, Permutability implies there exists a reversible channel $\mathscr{U}$ such that for all $i = 1,\ldots,n$ [26]



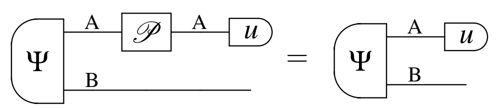
Applying the pure effect $\varphi_1^\dagger$ to both sides of the equation we obtain


(1)

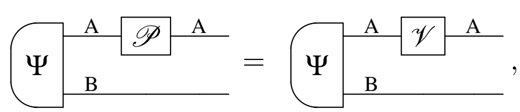
with



By construction, $\mathscr{P}$ is pure (by Purity Preservation) and occurs with probability 1 on all the states $\{\varphi_i\}_{i=1}^n$, consequently $\mathscr{P}$ is deterministic upon input of the state $\rho = \frac{1}{n}\sum_{i=1}^n \varphi_i$, that is $u\mathscr{P} =_\rho u$. Then, proposition 1 implies that for every purification $\Psi \in \mathsf{PurSt}_1(\mathrm{A}\otimes\mathrm{B})$ of ρ, one has



meaning that the pure states $(\mathscr{P}\otimes\mathscr{I}_\mathrm{B})\Psi$ and Ψ have the same marginal on system B. Hence, the uniqueness of purification implies that there exists a reversible transformation $\mathscr{V}$ such that



namely, by proposition 1, $\mathscr{P} =_\rho \mathscr{V}$. Combining this fact with Eq. (1), we obtain the condition $\mathscr{V}\varphi_i = \psi_i$ for every $i = 1,\ldots,n$. This proves that the states $\{\varphi_i\}_{i=1}^n$ can be reversibly transformed into (a subset of) the states $\{\psi_j\}_{j=1}^m$. Since both sets are maximal, we must have $n = m$. □

Permutability/Strong Symmetry implies that the eigenvalues of states are independent of the chosen diagonalisation [13]. Moreover, for every non-trivial system there exists a positive integer $d \geq 2$, called the *dimension* of the system, that fixes the cardinality of all maximal sets of perfectly distinguishable pure states.

Another structural result that is crucial for our work is that invariant states are stable under tensor product.

Proposition 4. *Let $\{\alpha_i\}_{i=1}^{d}$ be any maximal set of perfectly distinguishable pure states in a theory satisfying Causality, Purity Preservation, Purification, Pure Sharpness, and Permutability/Strong Symmetry. Then a state χ is invariant if and only if $\chi = \frac{1}{d}\sum_{i=1}^{d}\alpha_i$. Moreover, the perfectly distinguishing test for $\{\alpha_i\}_{i=1}^{d}$ is $\left\{\alpha_i^\dagger\right\}_{i=1}^{d}$.*

As a consequence, we have the desired result.

Proposition 5. *The invariant state of system* $\mathrm{A}\otimes\mathrm{B}$ *is the product of the invariant states of systems* A *and* B*:* $\chi_{\mathrm{AB}} = \chi_{\mathrm{A}}\otimes\chi_{\mathrm{B}}$.

Proof. We know that $\chi_{\mathrm{A}} = \frac{1}{d_{\mathrm{A}}}\sum_{i=1}^{d_{\mathrm{A}}}\alpha_i$ and $\chi_{\mathrm{B}} = \frac{1}{d_{\mathrm{B}}}\sum_{j=1}^{d_{\mathrm{B}}}\beta_j$, where $\{\alpha_i\}_{i=1}^{d_{\mathrm{A}}}$ and $\{\beta_j\}_{j=1}^{d_{\mathrm{A}}}$ are maximal sets of perfectly distinguishable pure states of systems A and B respectively. Let us consider the product state $\chi_{\mathrm{A}}\otimes\chi_{\mathrm{B}}$; it can be diagonalised as

$$\chi_{\mathrm{A}}\otimes\chi_{\mathrm{B}} = \frac{1}{d_{\mathrm{A}}d_{\mathrm{B}}}\sum_{i=1}^{d_{\mathrm{A}}}\sum_{j=1}^{d_{\mathrm{B}}}\alpha_i\otimes\beta_j,$$

The perfectly distinguishing test is $\left\{\alpha_i^\dagger\otimes\beta_j^\dagger\right\}$, so the set $\left\{\alpha_i\otimes\beta_j\right\}$ is maximal, otherwise the perfectly distinguishing test could not be made of all normalised pure effects, as it is straightforward to check. By proposition 4,

$$\frac{1}{d_{\mathrm{A}}d_{\mathrm{B}}}\sum_{i=1}^{d_{\mathrm{A}}}\sum_{j=1}^{d_{\mathrm{B}}}\alpha_i\otimes\beta_j = \chi_{\mathrm{AB}} = \chi_{\mathrm{A}}\otimes\chi_{\mathrm{B}}.$$

□

This also implies that the dimension of a composite system is the product of the dimensions of the components (Hardy's information locality [25, 26]): $d_{\mathrm{AB}} = d_{\mathrm{A}}d_{\mathrm{B}}$.

The fact that invariant states are stable under tensor products is an essential condition for every resource theory that regards invariant states as *free*. Indeed, a fundamental requirement in every resource theory is that the product of two free states be a free state [18, 19].

4 Resource theories of purity

Here we present three possible definitions of a resource theory of purity, based on three different choices of free operations.

4.1 Resource theory of random reversible channels

The most direct way of defining a resource theory of purity in general probabilistic theories is to set the free operations to be random reversible channels [12]:

Definition 3. A *random reversible (RaRe) channel*, is a channel $\mathscr{R}$ of the form $\mathscr{R} = \sum_i p_i\mathscr{U}_i$, where $\{p_i\}$ is a probability distribution, and $\mathscr{U}_i$ is a reversible channel for every i.

Using RaRe channels as free operations we can define a preorder between states [12]:

Definition 4. Let ρ and σ be normalised states. ρ is *purer* than σ if there exists a RaRe channel $\mathscr{R}$ such that $\sigma = \mathscr{R}\rho$. If ρ is purer than σ and σ is purer than ρ, we say that they are *equally pure*.

The same preorder was independently proposed by Müller and Masanes in the context of communicating spatial directions [34].

Note that the above resource theory of purity does not require any axioms at all—except, of course, the axioms encapsulated in the definition of general probabilistic theories. Moreover, the notion of purity as a resource does not rely on any pre-established notion of invariant state. Thanks to this fact, the definition can be applied to scenarios where the invariant state does not exist (e.g. for certain infinite-dimensional systems) and to settings where there exist more than a single invariant state [12].

In the following we will consider two alternative resource theories of purity, which build the free operations from the invariant state. These resource theories require more structure than the theory of RaRe channels. For example, in order to regard the invariant state as a free state, one needs to guarantee that the product of two invariant states is itself invariant [18, 19]—a non-trivial property that may fail in some theories. In order to guarantee the validity of this and other useful properties, in the following we will always assume the axioms of Causality, Purity Preservation, Purification, Pure Sharpness, and Permutability/Strong Symmetry.

4.2 The resource theory of unital channels

The broadest notion of free operations in a resource theory is the set of operations that preserve the free states. Regarding invariant states as free states, it is natural to consider a resource theory where the free operations are *unital channels*:

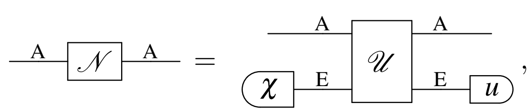
Definition 5. A channel $\mathscr{D}$ on system A is called *unital* if $\mathscr{D}\chi = \chi$, where χ is the invariant state of system A.

Clearly, reversible channels and RaRe channels are examples of unital channels. However, the converse is not true in general: for example, in quantum theory there exist unital channels that are not random unitary [30].

4.3 The resource theory of noisy operations

Another way to construct a resource theory of purity is by assuming noisy operations as free [27, 21], i.e. the operations generated by preparing invariant states, applying reversible channels, and discarding some of the outputs:

Definition 6. A *basic noisy operation* $\mathscr{N}$ on system A is a channel that can be decomposed as



(2)

for some suitable system E and some reversible channel $\mathscr{U}$. The set of *noisy operations* is the topological closure of the set of basic noisy operations.

The closure is needed because basic noisy operations do not form a closed set [37].

With the above definition, one has the obvious inclusion

Proposition 6. *Noisy operations are unital channels.*

Proof. By definition, one has

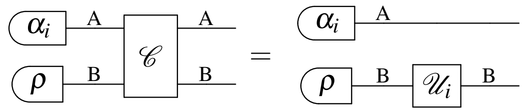
$$\mathscr{N}\chi_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{E}}\mathscr{U}_{\mathrm{AE}}\left(\chi_{\mathrm{A}} \otimes \chi_{\mathrm{E}}\right) = \mathrm{Tr}_{\mathrm{E}}\left(\chi_{\mathrm{A}} \otimes \chi_{\mathrm{E}}\right) = \chi_{\mathrm{A}}, \tag{3}$$

where we have used the fact that $\chi_{\mathrm{A}} \otimes \chi_{\mathrm{E}}$ is the invariant state of system $\mathrm{A} \otimes \mathrm{E}$ (by proposition 5). Clearly, every limit of operations satisfying Eq. (3) will also satisfy the same condition. □

It is known that in general noisy operations form in fact a *strict* subset of unital channels [22].

We will now prove that RaRe channels are a subset of noisy operations. To do that we use the notion of controlled reversible channels, recently introduced by Lee and Selby [31]:

Definition 7. Consider a maximal set of perfectly distinguishable pure states $\{\alpha_i\}_{i=1}^{d}$ of system A, and a set of reversible transformations $\{\mathscr{U}_i\}$, where the $\mathscr{U}_i$'s need not be all distinct, on another system B. A *controlled reversible channel* with control A and target B is a channel $\mathscr{C} \in \mathsf{Transf}(\mathrm{A}\otimes\mathrm{B})$ such that



In Ref. [31] it was shown that for theories satisfying Causality, Purification, Strong Symmetry, admitting perfectly distinguishable pure states, and such that the product of pure states is pure, controlled reversible channels are reversible on the combined system $\mathrm{A}\otimes\mathrm{B}$. Now we can use them to show that every RaRe channel is a noisy operation. The proof will be virtually the same as the one appeared in Ref. [21], clearly adapted to our case though, but the mathematical argument will be made more precise and rigorous.

Proposition 7. *Every RaRe channel is a noisy operation.*

Proof. Consider a RaRe channel $\mathscr{R} = \sum_{i=1}^{m} p_i \mathscr{U}_i$ on system A, and take an ancillary system E with dimension N. Let $\mathsf{S} = \left\{\varphi_j\right\}_{j=1}^{N}$ be a maximal set of perfectly distinguishable pure states of system E. Now, consider $m+1$ *disjoint* subsets S_i (for $i = 1,\dots,m+1$) of S, such that $|\mathsf{S}_i| = [p_i N]$ for $i = 1,\dots,m$, where $[p_i N]$ is the integral part[3] of $p_i N$, and $\mathsf{S} = \bigcup_{i=1}^{m+1} \mathsf{S}_i$. We wish to define a controlled reversible channel $\mathscr{C}$ on $\mathrm{A}\otimes\mathrm{E}$, with control E and target A, such that

$$\mathscr{C}\left(\rho\otimes\varphi_j\right) = \begin{cases} \mathscr{U}_i\rho\otimes\varphi_j & \text{if } \varphi_j \in \mathsf{S}_i \text{ for all } i = 1,\dots,m \\ \rho\otimes\varphi_j & \text{if } \varphi_j \in \mathsf{S}_{m+1} \end{cases}, \tag{4}$$

for all $\rho \in \mathsf{St}_1(\mathrm{A})$. Here the set of reversible channels applied to A is $\{\mathscr{U}_i\}_{i=1}^{m+1}$, where we set $\mathscr{U}_{m+1}$ to be the identity $\mathscr{I}$. We know that $\mathscr{C}$ is a reversible channel on $\mathrm{A}\otimes\mathrm{E}$ [31], therefore we can define the basic noisy operation

$$\mathscr{N}\rho = \mathrm{Tr}_{\mathrm{E}}\mathscr{C}\left(\rho\otimes\chi_{\mathrm{E}}\right),$$

where $\mathscr{C}$ is given by Eq. (4), and χ_{E} is the invariant state of system E. By Eq. (4)

$$\mathscr{N}\rho = \frac{1}{N}\sum_{j=1}^{N}\mathrm{Tr}_{\mathrm{E}}\mathscr{C}\left(\rho\otimes\varphi_j\right) = \frac{1}{N}\sum_{j=1}^{N}\mathrm{Tr}_{\mathrm{E}}\left(\mathscr{U}_i\rho\otimes\varphi_j\right) = \frac{1}{N}\sum_{i=1}^{m+1}|\mathsf{S}_i|\,\mathscr{U}_i\rho.$$

This is a RaRe channel $\mathscr{R}'$ arising a mixture of $m+1$ reversible channels $\{\mathscr{U}_i\}_{i=1}^{m+1}$ with probability distribution $\left\{\frac{|\mathsf{S}_i|}{N}\right\}$, which is not quite the original RaRe channel $\mathscr{R}$. Now we will show that in the limit of large N we get $\mathscr{R}$. This would mean that $\mathscr{R}$ could be arbitrarily well approximated by basic noisy operations, therefore it would be a noisy operation itself.

The first step is to prove that we can get rid of $\mathscr{U}_{m+1} = \mathscr{I}$. Recall that $|\mathsf{S}_i| = [p_i N]$ for $i = 1,\dots,m$, consequently

$$N = \sum_{i=1}^{m}[p_i N] + |\mathsf{S}_{m+1}|.$$

[3] Recall the integral part of a real number is defined as $[x] := \max_{n\in\mathbb{Z}}\{n \le x\}$.

By definition of integral part $0 \leq p_iN - [p_iN] \leq 1$, so by summing over i, for $i = 1, \ldots, m$, we get $0 \leq |\mathsf{S}_{m+1}| \leq m$, which implies

$$0 \leq \frac{|\mathsf{S}_{m+1}|}{N} \leq \frac{m}{N}.$$

Therefore in the limit of $N \to +\infty$, the term $\frac{|\mathsf{S}_{m+1}|}{N}$ vanishes. Now we want to prove that for $i = 1, \ldots, m$, the terms $\frac{[p_iN]}{N}$ converge to p_i in the limit of $N \to +\infty$. As $p_iN - 1 \leq [p_iN] \leq p_iN$, we have

$$p_i - \frac{1}{N} \leq \frac{[p_iN]}{N} \leq p_i,$$

which proves that $\frac{[p_iN]}{N}$ converges to p_i when $N \to +\infty$. This concludes the proof. □

Also in this case, the inclusion is strict: there exist noisy operations that are not RaRe channels [37]. In summary, we have the strict inclusions

$$\text{RaRe} \subset \text{noisy} \subset \text{unital}. \tag{5}$$

5 The majorisation criterion

Let us recall the mathematical notion of majorisation:

Definition 8. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. If $x_{[i]}$ denotes the i-th entry of the decreasing rearrangement of $\mathbf{x}$, we say that $\mathbf{x}$ *is majorised* by $\mathbf{y}$ (or that $\mathbf{y}$ *majorises* $\mathbf{x}$) if

- $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, \ldots, d-1$
- $\sum_{i=1}^{d} x_{[i]} = \sum_{i=1}^{d} y_{[i]}$.

Another useful characterisation of majorisation is in terms of doubly stochastic matrices: $\mathbf{x} \preceq \mathbf{y}$ if and only if $\mathbf{x} = D\mathbf{y}$, where D is a doubly stochastic matrix [23, 32].

For probabilistic theories where states can be diagonalised, majorisation allows one to define a preorder between states [13, 2, 29].

5.1 Majorisation and RaRe channels

In our previous work [13] we showed that, under the validity of the five axioms presented before, the majorisation preorder is equivalent to the preorder induced by considering RaRe channels as free operations:

Theorem 1. *Let ρ and σ be normalised states, and let $\mathbf{p}$ and $\mathbf{q}$ be the vectors of their eigenvalues respectively. Then ρ can be converted into σ by a RaRe channel if and only if $\mathbf{q} \preceq \mathbf{p}$.*

See theorems 4 and 5 of Ref. [13] for the proof. It is worth clarifying the role of Permutability/Strong Symmetry in the context of the other axioms:

Proposition 8. *In a theory satisfying Causality, Purity Preservation, Purification, and Pure Sharpness, the following are equivalent.*

1. *The theory satisfies Permutability/Strong Symmetry*
2. *The majorisation condition is sufficient for the convertibility of states under RaRe channels.*

Proof. We already know that 1 implies 2. Therefore, we need to show only the converse implication. Suppose the majorisation condition is sufficient for RaRe convertibility. Then, consider two states ρ and σ diagonalised as $\rho = \sum_{i=1}^{d} p_i \alpha_i$, and $\sigma = \sum_{i=1}^{d} p_i \alpha'_i$, with $p_1 \geq p_2 \geq \ldots \geq p_d > 0$. Since the two states have the same eigenvalues, the sufficiency of the majorisation condition (theorem 1) implies that they are equally pure. In Ref. [12] we noted that if two states are equally pure, there exists a reversible channel $\mathscr{U}$, such that $\sigma = \mathscr{U}\rho$, as already proved in Ref. [34]. Applying the effect $\alpha_1'^{\dagger}$ to both sides of this equality, we obtain

$$p_1 = \left(\alpha_1'^{\dagger}|\sigma\right) = \sum_j p_j \left(\alpha_1'^{\dagger}\middle|\mathscr{U}\,|\alpha_j\right) = \sum_j T_{1j} p_j \leq p_1,$$

having used the fact that the transition matrix $T_{ij} := \left(\alpha_i'^{\dagger}\middle|\mathscr{U}\,|\alpha_j\right)$ is doubly stochastic (see lemma 4 of Ref. [13]). The above condition is satisfied only if $\left(\alpha_1'^{\dagger}\middle|\mathscr{U}\,|\alpha_1\right) = 1$, that is, by proposition 2, only if $\mathscr{U}\alpha_1 = \alpha'_1$. Now consider the states $\rho_1 = \sum_{i=2}^{d} p_i \alpha_i$, and $\sigma_1 = \sum_{i=2}^{d} p_i \alpha'_i$ Repeating the previous argument, this time for p_2, now we can show the equality $\mathscr{U}\alpha_2 = \alpha'_2$. Iterating the procedure d times, we finally obtain the desired equality $\mathscr{U}\alpha_i = \alpha'_i$ for every i. Hence, every two maximal sets of perfectly distinguishable pure states are connected by a reversible channel. □

5.2 Majorisation and unital channels

Under the validity of our axioms, the preorder defined by unital channels coincides with the preorder defined by RaRe channels:

Proposition 9. *Let ρ and σ be two normalised states. ρ can be converted into σ by a RaRe channel if and only if ρ can be converted into σ by a unital channel.*

Proof. Necessity follows from the fact that RaRe channels are unital. Let us prove sufficiency. Let $\rho = \sum_{j=1}^{d} p_j \alpha_j$ and $\sigma = \sum_{j=1}^{d} q_j \alpha'_j$ be diagonalisations of ρ and σ respectively. Suppose we know that $\sigma = \mathscr{D}\rho$, where $\mathscr{D}$ is a unital channel. Then

$$\sum_{j=1}^{d} q_j \alpha'_j = \sum_{j=1}^{d} p_j \mathscr{D}\alpha_j.$$

Let us apply $\alpha_i'^{\dagger}$ to both sides, obtaining

$$q_i = \sum_{j=1}^{d} p_j \left(\alpha_i'^{\dagger}\middle|\mathscr{D}\,|\alpha_j\right).$$

This expression can be rewritten as $q_i = \sum_{j=1}^{d} D_{ij} p_j$, where $D_{ij} := \left(\alpha_i'^{\dagger}\middle|\mathscr{D}\,|\alpha_j\right)$. Let us prove that the D_{ij}'s are the entries of a doubly stochastic matrix D. Clearly $\left(\alpha_i'^{\dagger}\middle|\mathscr{D}\,|\alpha_j\right) \geq 0$ for all $i, j \in \{1, \ldots, d\}$ because $\left(\alpha_i'^{\dagger}\middle|\mathscr{D}\,|\alpha_j\right)$ is a probability. Let us calculate $\sum_{i=1}^{d} \left(\alpha_i'^{\dagger}\middle|\mathscr{D}\,|\alpha_j\right)$. We know that $\left\{\alpha_i'^{\dagger}\right\}_{i=1}^{d}$ is an observation-test and therefore it is normalised to the deterministic effect u.

$$\sum_{i=1}^{d} \left(\alpha_i'^{\dagger}\middle|\mathscr{D}\,|\alpha_j\right) = (u|\,\mathscr{D}\,|\alpha_j).$$

Now, $\mathscr{D}$ is a channel, therefore $u\mathscr{D} = u$ by Causality, then

$$\sum_{i=1}^{d} \left(\alpha_i'^{\dagger} \middle| \mathscr{D} \middle| \alpha_j \right) = \operatorname{Tr} \alpha_j = 1$$

because the states α_j's are normalised. Now let us calculate $\sum_{j=1}^{d} \left(\alpha_i'^{\dagger} \middle| \mathscr{D} \middle| \alpha_j \right)$. By proposition 4,

$$\sum_{j=1}^{d} \left(\alpha_i^{\dagger} \middle| \mathscr{D} \middle| \alpha_j' \right) = d \left(\alpha_i^{\dagger} \middle| \mathscr{D} \middle| \chi \right) = d \left(\alpha_i^{\dagger} \middle| \chi \right) = d \cdot \frac{1}{d} = 1,$$

where we have used the fact that unital channels leave χ invariant. Now if $\mathbf{p}$ is the vector of the eigenvalues of ρ, and $\mathbf{q}$ is the vector of the eigenvalues of σ, we have $\mathbf{q} = D\mathbf{p}$, and therefore $\mathbf{q} \preceq \mathbf{p}$ because D is doubly stochastic. By theorem 1 there exists a RaRe channel $\mathscr{R}$ such that $\sigma = \mathscr{R}\rho$. □

As a consequence, the majorisation criterion characterises both the preorder induced by RaRe channels and the preorder induced by *all* unital channels.

5.3 Equivalence of the three resource theories of purity

Summing up the previous results, we have shown that in a theory satisfying Causality, Purity Preservation, Purification, Pure Sharpness and Permutability/Strong Symmetry the three natural notions of purity as a resource coincide. Specifically, we have the following:

Theorem 2. *In a theory satisfying Causality, Purity Preservation, Purification, Pure Sharpness and Permutability/Strong Symmetry, for all states ρ and σ the following are equivalent.*

1. *ρ can be transformed into σ by a RaRe channel.*
2. *ρ can be transformed into σ by a noisy operation.*
3. *ρ can be transformed into σ by a unital channel.*
4. *The eigenvalues of ρ majorise the eigenvalues of σ.*

Proof. Straightforward from the inclusions (5) and proposition 9. □

6 Conclusions

In this work we have explored different ways of defining a resource theory of purity in general probabilistic theories. One definition adopts RaRe channels as free operations [12]. This definition has the advantage that it can be given in every theory, without the need of imposing additional structure. However, the most interesting features arise in theories that satisfy non-trivial axioms. For example, axioms like Purification and Purity Preservation imply a duality between the resource theory of purity and the resource theory of entanglement [12], and the further addition of Pure Sharpness guarantees that every state can be diagonalised [13]. Within this axiomatic scheme, we have shown that the purity preorder coincides with the majorisation preorder if and only if the theory satisfies the Strong Symmetry/Permutability axiom. The reason of the slash is that, in the context of our axioms, Strong Symmetry and Permutability are equivalent axioms. Specifically, Strong Symmetry/Permutability is inescapable if we want majorisation to be a *sufficient* criterion for the purity preorder. Furthermore, we have shown that the combination of our axioms with Permutability/Strong Symmetry guarantees a sort of universality in the definition of the purity preorder: when these axioms hold, the resource theories of purity defined via RaRe channels, unital channels, and noisy operations are all equivalent to the majorisation preorder.

Acknowledgements This work was supported by the Foundational Questions Institute (FQXi-RFP3-1325), by the National Natural Science Foundation of China (grants 11450110096 and 11350110207), and by the 1000 Youth Fellowship Program of China. CMS acknowledges the support by EPSRC doctoral training grant and by Oxford-Google Deepmind Graduate Scholarship. CMS wishes to thank Jonathan Barrett and Matty Hoban for interesting and inspiring discussions, and Stefano Gogioso for useful suggestions. CMS acknowledges insightful conversations with Lucien Hardy at Perimeter Institute for Theoretical Physics, which are gratefully acknowledged for their hospitality.

References

[1] S. Abramsky & B. Coecke (2004): *A categorical semantics of quantum protocols*. In: *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*, pp. 415–425, doi:10.1109/LICS.2004.1319636.

[2] H. Barnum, J. Barrett, M. Krumm & M. P. Müller (2015): *Entropy, majorization and thermodynamics in general probabilistic theories*. In C. Heunen, P. Selinger & J. Vicary, editors: Proceedings of the 12th International Workshop on *Quantum Physics and Logic*, Oxford, U.K., July 15-17, 2015, *Electronic Proceedings in Theoretical Computer Science* 195, pp. 43–58, doi:10.4204/EPTCS.195.4.

[3] H. Barnum, J. Barrett, M. Leifer & A. Wilce (2007): *Generalized No-Broadcasting Theorem*. *Phys. Rev. Lett.* 99, p. 240501, doi:10.1103/PhysRevLett.99.240501.

[4] H. Barnum, M. P. Müller & C. Ududec (2014): *Higher-order interference and single-system postulates characterizing quantum theory*. *New Journal of Physics* 16(12), p. 123029, doi:10.1088/1367-2630/16/12/123029.

[5] J. Barrett (2007): *Information processing in generalized probabilistic theories*. *Phys. Rev. A* 75, p. 032304, doi:10.1103/PhysRevA.75.032304.

[6] F. G. S. L. Brandão & G. Gour (2015): *Reversible Framework for Quantum Resource Theories*. *Phys. Rev. Lett.* 115, p. 070503, doi:10.1103/PhysRevLett.115.070503.

[7] G. Chiribella (2014): *Dilation of states and processes in operational-probabilistic theories*. In B. Coecke, I. Hasuo & P. Panangaden, editors: Proceedings 11th workshop on *Quantum Physics and Logic*, Kyoto, Japan, 4-6th June 2014, *Electronic Proceedings in Theoretical Computer Science* 172, pp. 1–14, doi:10.4204/EPTCS.172.1.

[8] G. Chiribella, G. M. D'Ariano & P. Perinotti (2010): *Probabilistic theories with purification*. *Phys. Rev. A* 81, p. 062348, doi:10.1103/PhysRevA.81.062348.

[9] G. Chiribella, G. M. D'Ariano & P. Perinotti (2011): *Informational derivation of quantum theory*. *Phys. Rev. A* 84, p. 012311, doi:10.1103/PhysRevA.84.012311.

[10] G. Chiribella, G. M. D'Ariano & P. Perinotti (2016): *Quantum Theory: Informational Foundations and Foils*, chapter Quantum from Principles, pp. 171–221. Springer Netherlands, Dordrecht, doi:10.1007/978-94-017-7303-4_6.

[11] G. Chiribella & C. M. Scandolo (2015): *Conservation of information and the foundations of quantum mechanics*. *EPJ Web of Conferences* 95, p. 03003, doi:10.1051/epjconf/20149503003.

[12] G. Chiribella & C. M. Scandolo (2015): *Entanglement and thermodynamics in general probabilistic theories*. *New Journal of Physics* 17(10), p. 103027, doi:10.1088/1367-2630/17/10/103027.

[13] G. Chiribella & C. M. Scandolo (2015): *Operational axioms for diagonalizing states*. In C. Heunen, P. Selinger & J. Vicary, editors: Proceedings of the 12th International Workshop on *Quantum Physics and Logic*, Oxford, U.K., July 15-17, 2015, *Electronic Proceedings in Theoretical Computer Science* 195, pp. 96–115, doi:10.4204/EPTCS.195.8.

[14] G. Chiribella & R. W. Spekkens, editors (2016): *Quantum Theory: Informational Foundations and Foils*. *Fundamental Theories of Physics* 181, Springer Netherlands, Dordrecht, doi:10.1007/978-94-017-7303-4.

[15] B. Coecke (2006): *Kindergarten Quantum Mechanics: Lecture Notes*. AIP Conference Proceedings 810(1), pp. 81–98, doi:10.1063/1.2158713.

[16] B. Coecke (2010): *Quantum picturalism*. Contemporary Physics 51, pp. 59–83, doi:10.1080/00107510903257624.

[17] B. Coecke, R. Duncan, A. Kissinger & Q. Wang (2016): *Quantum Theory: Informational Foundations and Foils*, chapter Generalised Compositional Theories and Diagrammatic Reasoning, pp. 309–366. Springer Netherlands, Dordrecht, doi:10.1007/978-94-017-7303-4_10.

[18] B. Coecke, T. Fritz & R. W. Spekkens (2014): *A mathematical theory of resources*. arXiv:1409.5531 [quant-ph].

[19] T. Fritz (2016): *Resource convertibility and ordered commutative monoids*. Mathematical Structures in Computer Science FirstView, pp. 1–89, doi:10.1017/S0960129515000444.

[20] J. Goold, M. Huber, A. Riera, L. del Rio & P. Skrzypczyk (2016): *The role of quantum information in thermodynamics—a topical review*. Journal of Physics A: Mathematical and Theoretical 49(14), p. 143001, doi:10.1088/1751-8113/49/14/143001.

[21] G. Gour, M. P. Müller, V. Narasimhachar, R. W. Spekkens & N. Yunger Halpern (2015): *The resource theory of informational nonequilibrium in thermodynamics*. Physics Reports 583, pp. 1–58, doi:10.1016/j.physrep.2015.04.003.

[22] U. Haagerup & M. Musat (2011): *Factorization and Dilation Problems for Completely Positive Maps on von Neumann Algebras*. Communications in Mathematical Physics 303(2), pp. 555–594, doi:10.1007/s00220-011-1216-y.

[23] G. H. Hardy, J. E. Littlewood & G. Pólya (1929): *Some simple inequalities satisfied by convex functions*. Messenger Math 58(145–152), p. 310.

[24] L. Hardy (2001): *Quantum theory from five reasonable axioms*. arXiv quant-ph/0101012.

[25] L. Hardy (2011): *Reformulating and reconstructing quantum theory*. arXiv:1104.2066 [quant-ph].

[26] L. Hardy (2016): *Quantum Theory: Informational Foundations and Foils*, chapter Reconstructing Quantum Theory, pp. 223–248. Springer Netherlands, Dordrecht, doi:10.1007/978-94-017-7303-4_7.

[27] M. Horodecki, P. Horodecki & J. Oppenheim (2003): *Reversible transformations from pure to mixed states and the unique measure of information*. Phys. Rev. A 67, p. 062104, doi:10.1103/PhysRevA.67.062104.

[28] M. Horodecki & J. Oppenheim (2013): *(Quantumness in the context of) resource theories*. International Journal of Modern Physics B 27(01n03), p. 1345019, doi:10.1142/S0217979213450197.

[29] M. Krumm (2015): *Thermodynamics and the Structure of Quantum Theory as a Generalized Probabilistic Theory*. arXiv:1508.03299 [quant-ph]. Master's thesis.

[30] L. J. Landau & R. F. Streater (1993): *On Birkhoff's theorem for doubly stochastic completely positive maps of matrix algebras*. Linear Algebra and its Applications 193, pp. 107–127, doi:10.1016/0024-3795(93)90274-R.

[31] C. M. Lee & J. H. Selby (2016): *Generalised phase kick-back: the structure of computational algorithms from physical principles*. New Journal of Physics 18(3), p. 033023, doi:10.1088/1367-2630/18/3/033023.

[32] A. W. Marshall, I. Olkin & B. C. Arnold (2011): *Inequalities: Theory of Majorization and Its Applications*. Springer Series in Statistics, Springer, New York, doi:10.1007/978-0-387-68276-1.

[33] C. B. Mendl & M. M. Wolf (2009): *Unital Quantum Channels – Convex Structure and Revivals of Birkhoff's Theorem*. Communications in Mathematical Physics 289(3), pp. 1057–1086, doi:10.1007/s00220-009-0824-2.

[34] M. P. Müller & L. Masanes (2013): *Three-dimensionality of space and the quantum bit: an information-theoretic approach*. New Journal of Physics 15(5), p. 053040, doi:10.1088/1367-2630/15/5/053040.

[35] C. Piron (1976): *Foundations of quantum physics*. Mathematical Physics Monograph Series, Benjamin-Cummings Publishing Company.

[36] P. Selinger (2011): *A Survey of Graphical Languages for Monoidal Categories*. In B. Coecke, editor: *New Structures for Physics*, *Lecture Notes in Physics* 813, Springer, Berlin, Heidelberg, pp. 289–356, doi:10.1007/978-3-642-12821-9_4.

[37] P. W. Shor (2010): *Structure of Unital Maps and the Asymptotic Quantum Birkhoff Conjecture*. presentation.

[38] A. Uhlmann (1971): *Sätze über Dichtematrizen*. *Wiss. Z. Karl-Marx Univ. Leipzig* 20, pp. 633–637.

[39] A. Uhlmann (1972): *Endlich-dimensionale Dichtematrizen I*. *Wiss. Z. Karl-Marx Univ. Leipzig.* 21, pp. 421–452.

[40] A. Uhlmann (1973): *Endlich-dimensionale Dichtematrizen II*. *Wiss. Z. Karl-Marx Univ. Leipzig* 22, pp. 139–177.