Interacting Frobenius Algebras are Hopf

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Theories featuring the interaction between a Frobenius algebra and a Hopf algebra have recently appeared in several areas in computer science: concurrent programming, control theory, and quantum computing, among others. Bonchi, Sobocinski, and Zanasi [10] have shown that, given a suitable distributive law, a pair of Hopf algebras forms two Frobenius algebras. Here we take the opposite approach, and show that interacting Frobenius algebras form Hopf algebras. We generalise [10] by including non-trivial dynamics of the underlying object—the so-called phase group—and investigate the effects of finite dimensionality of the underlying model. We recover the system of Bonchi et al as a subtheory in the prime power dimensional case, but the more general theory does not arise from a distributive law.

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1 Summary

Frobenius algebras and bialgebras are structures which combine a monoid and a comonoid on a single underlying object. They have a long history[1] in group theory, but have applications in many other areas: natural language processing [36, 37], topological quantum field theory [30], game semantics [32], automata theory [41], and distributed computing [8], to name but a few.

In quantum computation, the bialgebraic interplay between two Frobenius algebras describes the behaviour of complementary observables [13, 15], a central concept in quantum theory. This interaction is the basis of the ZX-calculus, a formal language for quantum computation. Using these ideas, a significant fraction of finite dimensional quantum theory can be developed without reference to Hilbert spaces [3, 4, 5, 6, 16, 17, 18, 19, 40, 20, 21, 22, 23, 26, 27, 28, 29, 33, 34, 35, 39]. Surprisingly, almost exactly the same axioms have also appeared in totally different settings: Petri nets [12, 38] and control theory [7, 11]. This combination of structures seems to have broad relevance in computer science.

The approach of the current paper is directly inspired by the recent work of Bonchi, Sobociński, and Zanasi [10], who investigated the theory of interacting Hopf algebras[2] and showed that Hopf algebras which obey a certain distributive law form Frobenius algebras [9, 10]. Using Lack's technique of composing PROPs [31], they show the resulting theory $\mathbb{IH}_R$ is isomorphic to that of linear relations[3].

Do interacting quantum observables [15] admit such a beautiful description? In this paper we present a rational reconstruction of theory of strongly complementary observables and show that, except under quite restrictive circumstances, the theory does not arise by composing PROPs via a distributive law. Along the way we also clarify the structure of the theory of complementary observables and show that some assumptions used in earlier work are unnecessary.

[1] See Fauser [25] for much detail on Frobenius algebras, including their history; for the history of Hopf algebras see [2].
[2] A Hopf algebra is a bialgebra with some extra structure; see later **??**.
[3] Baez and Erbele [7] prove the same result with different techniques.

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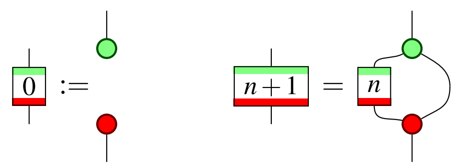
The starting point is the insight that an observable of some quantum system corresponds to a Frobenius algebra on its state space [14]. Further, the state spaces have non-trivial endomorphims giving their internal dynamics; among these there is a *phase group* for each observable, which leaves the observable unchanged. Since observables are fundamental to quantum theory, we take Frobenius algebras and their phase groups as the starting point, and freely construct $\mathbf{F}G$, the PROP of a Frobenius algebra with a given group of phases G.

We then consider a pair of such Frobenius algebras and formalise interactions between them by imposing stronger and stronger axioms upon them. We produce a series of PROPs

$$\mathbf{F}\_ + \mathbf{F}\_ \twoheadrightarrow \mathbf{IF} \twoheadrightarrow \mathbf{IFK} \twoheadrightarrow \mathbf{IFK_d}$$

each more closely resembling quantum theory than its predecessor. The first is simply the disjoint union of two *non*-interacting observables. The second requires that the observables be strongly complementary; this means their corresponding Frobenius algebras jointly form a Hopf algebra [13, 15]. The additional structure allows us to construct a ring of endomorphisms of the generator, distinct from the phase groups, and a large class of multiparty unitary operations, being the abstract counterpart of quantum circuits. Among these morphisms, the *internal integers* play an important role:



The next two PROPs introduce eigenstates for the observables, and the effect of finite dimensionality of the state space respectively. In the last of these, $\mathbf{IFK_d}$, if the dimension is a prime power then we recover the system $\mathbb{IH}_R$ of Bonchi et al [10] as a subcategory. However we note that if the ring of internal integers has zero divisors, for example in 4-D quantum systems, then their axioms do not hold.

Each of these theories is actually a functor from a suitable category of groups, so we can freely construct a quantum-like theory with any given dynamics. It is these phase groups that ultimately obstruct the derivation of a distributive law of PROPS.

Theorem. *There is no distributive law of PROPs*

$$\tau : \mathbf{F}G;\mathbf{F}H \to \mathbf{F}H;\mathbf{F}G$$

which gives rise to $\mathbf{IF}(G,H)$.

Our motivation for studying these generalisations is to better understand categorical quantum theory [1], particularly with a view to the ZX-calculus. We explicate the necessary features of higher dimensional versions of the calculus, and separate the algebraic foundation from model-specific details. This will help clarify questions of completeness [3, 6, 24, 33] and also aid in the formalisation of error correcting codes [22]. However given the interest in these structures in other areas, we expect that a richer theory will lead to unexpected applications elsewhere. As a side-effect we learn that the theory of Petri-nets differs from quantum mechanics.

Comparison to ZX-calculus. The main inspiration for this approach is the ZX-calculus. Writing S^1 for the circle group, the PROP $\mathbf{IFK_d}(S^1,\mathbb{Z}_2,S^1,\mathbb{Z}_2)$, contains all the elements and most of the equations of the ZX-calculus, but there are some key differences. Firstly, S^1 is the entire phase group i.e. no new phases are generated by the action of $\mathbb{Z}_2$. Secondly, the ZX-calculus incorporates the *Hadamard gate*, which is a definable map which exchanges the colours. In consequence, the sets of - and -unbiased points are not disjoint in the ZX-calculus. We will explore necessary and sufficient conditions for such a map to exist abstractly in future work; a connection with Gogioso and Zeng's [27] seems likely.

Comparison to "Interacting Hopf algebras". The similarities between our system and that of Bonchi, Sobocinksi, and Zanasi [10] are striking. They consider the theory of interacting Hopf algebras.

Taking the Frobenius structure as primitive yields *almost* the same theory as starting with the Hopf structure, and requires fewer axioms to be imposed. The main extra ingredient in **IF** are the phase groups, which play rather badly with the Hopf algebra structure as the following Lemma shows:

Lemma. *Let α be a phase of either colour in* $\mathbf{IF}(G,H)$*; then α is bialgebra morphism iff* $\alpha = \mathrm{id}$.

Since the phases are not present in the systems considered by Bonchi et al, we temporarily restrict attention to $\mathbf{IF}(1,1)$, i.e. where both phase groups are trivial. Unlike in **IF**, all the PROPs of [10] have trivial scalars; this forces the generating object to be 1-dimensional.

The theory **IF** contains both **HA** and $\mathbf{HA}^{\mathrm{op}}$ studied by Bonchi et al; however it does not validate any of the axioms concerning the invertibility of the ring elements, nor their commutation with the "wrong" bialgebra maps[4]. Here Bonchi et al rely on the assumption that R is a PID. However this assumption fails in, e.g., $\mathbb{C}\mathbb{Z}_4$ which is a perfectly good model of **IF**. However, in prime dimensional models these axioms are validated, hence:

Theorem. *Every* $\mathbf{IFK_d}$ *algebra of prime dimension includes a copy of* $\mathbb{IH}_R^w$ *and this coincides with the image of* $\mathbf{IF}(1,1)$*, modulo scalar factors.*

Further work Many interesting algebraic properties of **IF** and its relatives remain unexplored: most notable is generation of new phases via the semi-direct product in the phase group, and the possibility to define Hadamard transforms purely abstractly. A tempting next phase of development would to investigate topological features, by considering the case of Lie groups. Finally we note that in the *models* of the ZX-calculus (although not derivable) we have the Euler decomposition for $\mathrm{SU}(2)$ giving every unitary as a composition of at most three unitaries. An abstract understanding of this would be most valuable.

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[4] to wit: (W1), (W7), (W8), (B1), (B7), (B8), (S1) and (S2).

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