Hypergraph states, their entanglement and robustness properties

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Abstract

We study nonlocal properties of hypergraph states, or nonlocal stabilizer states, which are generalisations of well-known graph states. We find that some families of hypergraph states violate local realism exponentially like GHZ states, but are more robust than their graph-state analogues. We give applications in quantum metrology and measurement-based quantum computation. Additionally, we extend the rule of local complementation for characterizing unitaries transformations. Finally, we discuss the computational complexity of stabilizing group of hypergraph states in the framework of monomial stabilizer formalism.

Introduction. — Studying multiparticle entanglement is crucial, however, its characterization is hindered by the exponentially increasing dimension of the Hilbert space. Therefore, we restrict ourselves to nonlocal properties of quantum hypergraph states [3], which are generalizations of the well-known graphs states, which themselves are very important in quantum information [1]. They are given by graphs or via a stabilizer formalism.

Hypergraph states are special cases of the locally maximally entangleable states [3]. They correspond to hypergraphs, i.e., a single hyperedge can connect more than two vertices. Hypergraph states are useful for quantum search algorithms, quantum fingerprinting protocols, are complex enough to serve as witnesses in all quantum-Merlin-Arthur problems and are investigated in condensed matter physics as ground states of spin models with interesting topological properties [4]. Despite their usefulness and elegant description, many of their important properties have not been extensively studied yet. In [8] we demonstrate new properties of these states.

Definitions. — A hypergraph $H = (V, E)$ consists of a set of N vertices and hyperedges $E \subset 2^V$, which can connect more than two vertices. It corresponds to the N-qubit hypergraph state $|H\rangle$:

$$|H\rangle = \prod_{e\in E} C_e \, |+\rangle^N , \tag{1}$$

where C_e is a multiqubit phase gate acting on the Hilbert space associated with the vertices $v \in e$ e, given by the matrix $C_e = \mathbb{1} - 2\,|11\ldots1\rangle\langle1\ldots11|$. The first nontrivial hypergraph state consists of $N = 3$ qubits [see Fig. 1 (a)]. A hyperedge has cardinality k, if it circumscribes k vertices and a hypergraph is k-uniform, if all edges have cardinality k. Alternatively, we can define the hypergraph state using a nonlocal stabilizer formalism. For each qubit i we define the operator (X denotes the Pauli matrix σ_x):

$$g_i = X_i \bigotimes_{e\in E} C_{e\setminus\{i\}}. \tag{2}$$

Then, similar to graph states, the hypergraph state can be defined as the unique eigenstate for all of them, $g_i\,|H\rangle = |H\rangle$ with the eigenvalue $+1$.

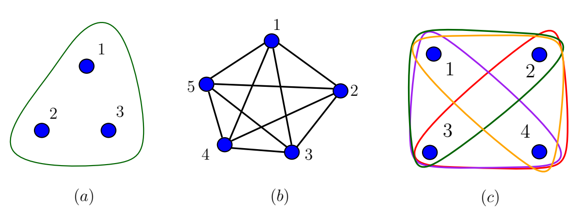


Figure 1. (a) The simplest HG state: $|H_3\rangle = C_{123}\,|+\rangle^{\otimes 3} = \frac{1}{\sqrt{8}}(|000\rangle + \cdots + |110\rangle - |111\rangle)$. (b) The GHZ state $|GHZ_5\rangle = \frac{1}{\sqrt{2}}(|00000\rangle + |11111\rangle)$. (c) The complete three-uniform HG.

Nonlocality tests, robustness and applications. — The key observation for the construction of our nonlocality arguments is that the stabilizer of hypergraph states, despite being nonlocal, predicts perfect correlations for some local measurements. Using this for the three-qubit state [see Fig. 1 (a)] we can write stabilizer operators: $g_1 = X_1 \otimes C_{23}, \quad g_2 = X_2 \otimes C_{13}, \quad g_3 = X_3 \otimes C_{12}$. We can expand the controlled phase gate C_{23} on two qubits, leading to $g_1 = X_1 \otimes (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|)$.
From this the perfect correlations follow: $P(+--|XZZ) = 0$ and $P(--+|XZZ) = P(-+-|XZZ) = P(-++|XZZ) = 0$ and if a fully LHV model satisfies these conditions and symmetric correlations coming from the permutations of the stabilizer operator, then it must fulfill that $P(+--|XXX) = 0$, when the actual value is $P(+--|XXX) = \frac{1}{16}$. The same holds for the permutations of the qubits. The above is a so-called Hardy argument. One can extend this result to bi-separable LHV models and derive a Bell inequality which is violated by the amount of $\frac{1}{16}$.
We extend our analysis to multipartite states starting with the generalization of the N-qubit GHZ state, which corresponds to the fully connected graph [see Fig. 1 (b) for $N = 5$]. Instead of having usual edges, we require every hyperedge to connect exactly three parties. Therefore, we get the complete three-uniform hypergraph state [see Fig. 1 (c) for $N = 4$]. It is known that GHZ states violate the Mermin inequalities with an amount that grows exponentially with the number of parties [2]. For a similar Bell operator $\mathcal{B}_N$ we find that:
Theorem 1. An N-qubit fully connected three-uniform hypergraph state violates local realism in Mermin-type inequalities by an amount growing exponentially with the number of qubits:

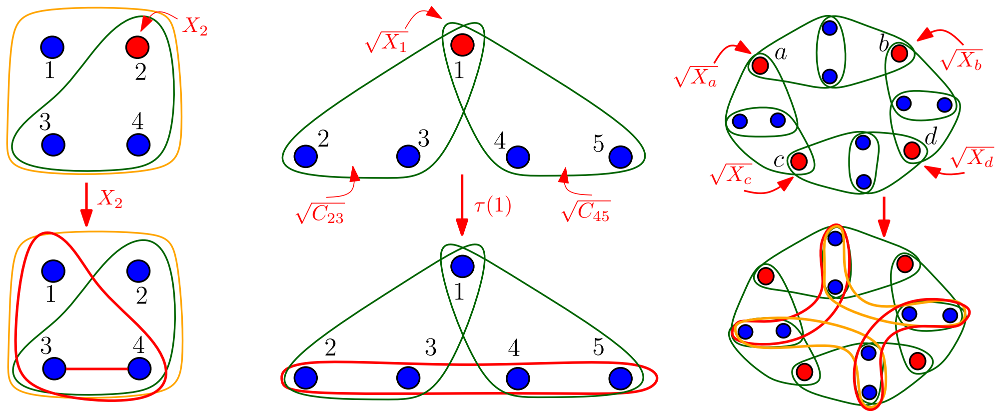
$$\langle \mathcal{B}_N \rangle_C = 2^{\lfloor N/2 \rfloor} \text{ - local HV models, } \langle \mathcal{B}_N \rangle_Q \geq 2^{N-2} - \frac{1}{2} \text{ - hypergraph states.} \tag{3}$$

We further extend our investigation to the complete four-uniform case and we find that the exponential violation is preserved. More precisely, the ratio between quantum and classical value is:

$$\frac{\langle \mathcal{B}_N \rangle_Q}{\langle \mathcal{B}_N \rangle_C} \overset{N\to\infty}{\sim} \frac{1.20711^N}{(2\sqrt{2}+2)}.$$

This shows that nonlocal stabilizer states are highly entangled and violate local realism in an extreme manner. Even more importantly, we find that they are more robust under particle loss than the GHZ states, which after tracing out one party become fully separable. The complete three- and four-uniform hypergraph states still violate the same inequalities after tracing out one or more particles. For some cases of complete four-uniform states, the reduced state shows the same exponential scaling of the Bell inequality violation as the original one. So, hypergraph states have a rich structure, deserving closer investigation.
Due to the extreme manner of violation, we find that complete three- and four-uniform hypergraph states can be used as a robust resource for Heisenberg-limited metrology and for a certain scheme of the measurement-based quantum computation. The robustness property, in addition, may allow the lower detection efficiency in the experiments.

Local complementation. — When talking about entanglement, it is crucial to develop the rules for local unitary equivalence classes of states. Such a rule in graph states have been long established already and is called local complementation (LC) [1]. Weaker rule under Pauli-X and -Z operations for hypergraphs was developed in [6], example is represented in Fig. 2 (a). Here we present a new rule for hypergraphs which serves in a very similar manner as LC in graph states:



(a) (b) (c)

Figure 2. (a) Pauli-X transformation; (b) Example of LC using square roots of Pauli-X and controlled-Z; (c) Example of LC using only square roots of Pauli-X.

Theorem 2. For any hypergraph state corresponding to a hypergraph $H = (V, E)$, we consider a transformation around a vertex $a \in V$, $\tau(a) = \sqrt{X_a} \prod_{e \in N(a)} \sqrt{C_e}$, which creates (removes) a new set of edges, which are the pairwise union of the edges containing a, with a removed.
Although a square root of controlled-Z gate,$\sqrt{C_e}$, is not a local gate, in particular structure of hypergraph states, the new rule can be chosen to be applied on multiple vertices in a way that nonlocal controlled-Z gates cancel each other. Therefore, the map is local and is done only by applying local $\sqrt{X_a}$'s. See Fig. 2 (c).

Discussion and Outlook — Establishing so-called LC rule between hypergraphs gives one more insight to which maps do not bring us out of hypergraph stabiliser formalism. More general stabiliser operators are discussed in [5] in the context of monomial stabilizer formalism. From the results of [5] it directly follows that the stabilizer unitaries of hypergraph states are classically simulatable, which means that classical efficient algorithms exist for important tasks like computing the expectation value of a k-qubit observable for some constant k. This is the first step to further investigate, if Gottesman-Knill theorem can be extended to nonlocal stabilisers as well, i.e. if nonlocal stabilizer circuits can be simulated efficiently on a classical computer.

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