Operational Theories of Physics as Categories (Overview)

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The advent of the field of quantum information has led to much interest in understanding the quantum world in terms of the *operations* it allows one to perform [8, 2, 4, 1]. This has culminated in several reconstructions of quantum theory from among more general operational theories, defined in terms of systems and probabilistic measurements one may perform upon them [9, 4]. A programme initiated by Abramsky and Coecke [1] has demonstrated the power of *category theory* as a tool for describing the basic features of such theories, and provided a fully categorical approach to the study of quantum theory itself. In the article [11], summarized here, we present a new categorical formalism for the study of more general operational theories of physics, connecting the probabilistic and categorical frameworks.

Operational theories with control

Let us first introduce our notion of an operational theory of physics, based closely on those of Chiribella, D'Ariano and Perinotti [3]. The most basic ingredients of these are physical *systems* $A,B,C,\dots$ and *events* $f\colon A \to B$ which may occur between them. Events $f\colon A \to B$ and $g\colon B \to C$ may be composed 'one after the other' to give a new event $g \circ f\colon A \to C$, and typically, as we always assume in this overview, we may also compose systems and events side-by-side as in $A \otimes B$, $f \otimes g$. This allows events to be described graphically using 'circuit diagrams', and is well-known to mean that they form a *symmetric monoidal category* [7], in particular coming with a distinguished system I representing 'nothing'.

Additionally, an operational theory specifies certain indexed collections of events $\{f_x\colon A \to B_x\}_{x\in X}$ called *tests*, with subcollections of these called *partial tests*. Each such test is thought of as a finite-outcome measurement one may perform on the system A. We imagine that, on any run of the test, one of the events f_x occurs, leaving a system B_x, with the *outcome* x from the finite set X then recorded.

This outcome data from tests may be used operationally in two main ways. The first, *coarse-graining*, allows one given a test of the form $\{f_x\colon A \to B\}_{x\in X} \cup \{g_y\}_{y\in Y}$ to 'merge' several of its outcomes to form a new test, denoted $\{\bigovee_{x\in X} f_x\} \cup \{g_y\}_{y\in Y}$. Mathematically, this provides a partially binary operation $f \ovee g\colon A \to B$ on events of each given type, with units $0_{A,B}\colon A \to B$ forming a family of *zero arrows*. The second, *control*, allows one given a test $\{f_x\colon A \to B_x\}_{x\in X}$, and for each outcome a test $\{g_{x,y}\colon B_x \to C_{x,y}\}_{y\in Y_x}$ to form a new test $\{g_{x,y} \circ f_x\}_{x\in X, y\in Y_x}$, intuitively with behaviour conditioned by the outcomes $x \in X$. Control is known to be closely related to a notion of *causal* structure in our theories, operationally corresponding to the presence of a unique *deterministic effect* $\dot{\top}_A$ on each system, referred to as *discarding* (see Lemmas 4 and 7 of [3]), i.e. a unique event $\dot{\top}_A\colon A \to I$ for which $\{\dot{\top}_A\}$ forms a test. In fact, causality follows from our final natural assumption, which states that each event f belongs to a unique test of the form $\{f\} \cup \{e\colon A \to I\}$. Intuitively, the effect e simply observes that 'f did not occur'.

We call such a theory Θ a *(monoidal) operational theory with control* (OTC). Examples include deterministic classical physics, in which events are described by partial functions between sets, and quantum theory, in which they are given by completely positive, sub-unital maps between Hilbert spaces or C*-algebras. Since we do not impose the typical assumption that *scalars*, i.e. events $p\colon I \to I$, correspond

*Supported by EPSRC Studentship OUCL/2014/SET

Submitted to:
QPL 2016

to actual probabilities $p \in [0,1]$, as in [4, 10], less intuitively physical examples are possible (see, for instance, §3.1 of [6]). Nonetheless OTCs allow for very similar reasoning to the theories of [4], such as the ability to form 'convex combinations' of physical events.

From theories to categories

Our goal is now to find a completely categorical treatment of these operational theories. From this perspective, it would be more natural to encode the outcome sets X 'in the objects'. Following [3, p.12], we capture this by saying that a theory has *direct sums* when each finite collection of systems $\{B_x\}_{x\in X}$ comes with a system $\bigoplus_{x\in X} B_x$ and test $\{\rhd_y \colon \bigoplus_{x\in X} B_x \to B_y\}_{y\in X}$ such that partial tests $\{f_x \colon A \to B_x\}_{x\in X}$ correspond simply to single events $f \colon A \to \bigoplus_{x\in X} B_x$ with $\rhd_x \circ f = f_x$ for all x. In fact, we may formally 'complete' any theory Θ to a new one Θ^+ coming with direct sums, and so it is natural to assume their presence. Explicitly, systems in Θ^+ are finite indexed collections $\{A_x\}_{x\in X}$ of systems of Θ, while events $M \colon \{A_x\}_{x\in X} \to \{B_y\}_{y\in Y}$ are 'matrices' of events $M(x,y) \colon A_x \to B_y$ whose 'columns' form partial tests.

Now, a crucial observation is that these direct sums may be described using a basic notion from category theory: they form finite *coproducts* $(+,0)$ in the category of events. By encoding the flow of outcome data from tests, these coproducts provide us with an succinct description of the theory. Partial tests $\{f_i \colon A \to B_i\}_{i=1}^n$ correspond simply to morphisms $f \colon A \to B_1 + \cdots + B_n$, with their individual events given by $f_j = \rhd_j \circ f$ where we may define canonical 'projections' $\rhd_j \colon A_1 + \cdots + A_n \to A_j$ thanks to the zero arrows $0_{A,B}$. Since each partial test is determined by its events, these $\rhd_j$ will together be *jointly monic*. The coproducts in fact inherently capture the ability to form controlled tests, and using the *codiagonal* morphisms $\nabla \colon B + \cdots + B \to B$ also describe coarse-graining via $\bigovee_{x\in X} f_x = \nabla \circ f$. Finally, the discarding effects ⏚ let us identify when a morphism f corresponds to a test categorically, namely when it satisfies $\text{⏚} \circ f = \text{⏚}$, and we call such f *total*. The main assumptions of an OTC are then captured by saying that each $f \colon A \to B$ is of the form $\rhd_1 \circ g$ for a unique total $g \colon A \to B + I$. Conversely, these categorical properties are in fact enough to define the full structure of an OTC, in the above way.

Summarizing, any OTC may be completed to one with direct sums, and then equivalently described as a symmetric monoidal category $\mathbf{C}$ (of events) with zero arrows and finite coproducts, along with a family of morphisms $\text{⏚}_A \colon A \to I$, satisfying the above properties as well as a couple of basic compatibility requirements. We call such a structure $(\mathbf{C}, \text{⏚})$ a *(monoidal) operational category in partial form*, our main example being the category $\mathbf{ParTest}(\Theta)$ of partial tests of an OTC Θ, i.e. of events in Θ^+.

Operational categories

Though we have reached a more concise description of our theory in terms of its category $\mathbf{C} = \mathbf{ParTest}(\Theta)$, our emphasis on *partial* tests may appear unnatural, and relies on the extra structure of discarding morphisms. For a more traditional categorical treatment, we finally restrict attention to the subcategory $\mathbf{B} = \mathbf{C}_{\text{total}} = \mathbf{Test}(\Theta)$ of total morphisms, i.e. the category of tests of Θ, or equivalently of deterministic events in Θ^+. Again $\mathbf{B}$ has coproducts, taken from $\mathbf{C}$, but causality now means that each object A has a unique morphism $A \to I$, making I a *terminal object* in $\mathbf{B}$. We summarize its properties as follows.

Definition. A *(monoidal) operational category* is a symmetric monoidal category $(\mathbf{B}, \otimes, I)$ with finite coproducts $(+,0)$ distributed over by the tensor, for which the tensor unit I is a terminal object 1, and:

- the morphisms $[\rhd_1, \kappa_2], [\rhd_2, \kappa_2] \colon (A+A)+1 \to A+1$ are jointly monic, where $\rhd_i \colon A + A \to A + 1$ are defined by $\rhd_1 = [\kappa_1, \kappa_2 \circ !]$ and $\rhd_2 = [\kappa_2 \circ !, \kappa_1]$;
- diagrams of the right-hand form are pullbacks.

$$\begin{tikzcd}
A \arrow[r, "!"] \arrow[d, "\kappa_1"'] & 1 \arrow[d, "\kappa_1"] \\
A+1 \arrow[r, "!+!"'] & 1+1
\end{tikzcd}$$

To make sense of the last two conditions in this definition, note that morphisms $f\colon A \to B+1$ in the category $\mathbf{B} = \mathbf{Test}(\Theta)$ of tests may be equated with morphisms $f\colon A \to B$ in the category $\mathbf{C}$ of partial tests, since each partial test $\{f_x\}_{x\in X}$ corresponds to a unique test of the form $\{f_x\}_{x\in X} \cup \{e\colon A \to I\}$, just as we previously noted for individual events. More generally, we call such morphisms in a suitable category $\mathbf{B}$ *partial morphisms* from A to B, and these form a new category $\mathrm{Par}(\mathbf{B})$ [5]. In this way the subcategory $\mathbf{B} = \mathbf{C}_{\text{total}}$ of tests in fact captures the full structure of $\mathbf{C} = \mathbf{ParTest}(\Theta) \simeq \mathrm{Par}(\mathbf{B})$ and hence the operational theory Θ^+. The conditions above simply assert basic properties of $\mathrm{Par}(\mathbf{B})$, ensuring this correspondence, which is summarized in our main result:

Theorem. The following structures are equivalent: i) a (monoidal) operational theory with control Θ with direct sums; ii) a (monoidal) operational category in partial form $\mathbf{C}$; iii) a (monoidal) operational category $\mathbf{B}$. The correspondences are $\mathbf{C} = \mathbf{Event}_\Theta$, $\Theta = \mathsf{OT}(\mathbf{C})$, $\mathbf{B} \simeq \mathbf{C}_{\text{total}}$ and $\mathbf{C} \simeq \mathrm{Par}(\mathbf{B})$.

For example, the OTCs corresponding to classical and quantum physics are described entirely by the respective operational categories $\mathbf{Set}$, of sets and functions, and $(\mathbf{FDim})\mathbf{CStar}^{\text{op}}_{\text{CPU}}$ of (finite-dimensional or arbitrary) C*-algebras and completely positive unital maps. Structurally, the above result, along with the passage from OTCs to those with direct sums, can be made precise in terms of (2-)functors between the (2-)categories of operational theories and categories.

Effectus theory

The categorical structures we have used were in fact largely first considered by Jacobs et al. [6] as part of a new area of categorical logic called *effectus theory*. A *(monoidal) effectus* is a (monoidal) operational category in which diagrams of the following form are pullbacks:

$$
\begin{tikzcd}
A \arrow[r, "!"] \arrow[d, "\kappa_1"'] & 1 \arrow[d, "\kappa_1"] \\
A+B \arrow[r, "!+!"'] & 1+1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A+B \arrow[r, "!+\mathrm{id}"] \arrow[d, "\mathrm{id}+!"'] & 1+B \arrow[d, "!+!"] \\
A+1 \arrow[r, "!+!"'] & 1+1
\end{tikzcd}
$$

Thanks to the above results, we can give a new operational interpretation to effectus theory, and reason about effectuses as operational theories satisfying these extra assumptions. Operationally, the first pullback corresponds to a basic requirement one might expect of any theory, namely that if $\{f_x\}_{x\in X}$ and $\{f_x\}_{x\in X} \cup \{g_y\}_{y\in Y}$ are both tests, then each event g_y is impossible and so equal to 0. In contrast, the second is a genuine, though mild, physical assumption allowing us to determine the general tests in our theory entirely from its *observations*: those consisting only of effects $\{e_x\colon A \to I\}_{x\in X}$. These assumptions are true of classical, probabilistic and quantum physics, and lead to nice behaviour in the theory, in particular with each space of effects forming an *effect algebra* [6, Proposition 17]. While effectus theory has been explicitly developed to describe each of these three forms of computation, these results suggest its use as a foundation for the study of computation in more general operational theories.

More broadly, the notion of an operational category can help provide new connections between the categorical and probabilistic approaches to the study of quantum foundations, by placing both in a common setting and allowing one to translate results about operational theories into a categorical language, and vice versa. A natural next step would now be to find properties which ensure that an operational category corresponds to a probabilistic theory in the sense of [3], and hence translate the existing quantum reconstruction theorems [4, 9] into a purely categorical form.

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