Ambiguity and Incomplete Information in Categorical Models of Language

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Categorical Compositional Semantics

Outline

- Meaning are modelled in dagger compact closed categories, typical examples are $\text{FdVect}, \text{CPM}(\text{FdVect}), \text{Rel}$
- Meanings of individual words are states in these compact closed categories, e.g. $I \xrightarrow{\text{dog}} N$
- How do we describe ambiguous and incomplete information in a systematic way?
Multiple Interpretations

When we encounter the noun “bank” in a sentence, questions arise:
Categorical Compositional Semantics

Ambiguity

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▶ Does the text intend a “river bank” or a “financial bank”?
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- Is one more likely than the other?
Multiple Interpretations

When we encounter the noun “bank” in a sentence, questions arise:

- Does the text intend a “river bank” or a “financial bank”?
- Is one more likely than the other?
- Given representations of “river bank” and “financial bank” can we build a model of “bank” as a mixture?

\[
0.9 \text{“river bank”} + 0.1 \text{“financial bank”}
\]
Move to $\text{CPM}(\text{FdHilb})$ and model ambiguity using density matrices:

$$0.9|r.b\rangle\langle r.b| + 0.1|f.b\rangle\langle f.b|$$
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Can we systematically use the CPM construction to provide a “mixing construction”? \( \text{CPM}(\text{Rel}) \) seems to suggest the answer is no:

- Strange mixing behaviour
- Poverty of scalars
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Do we care?
Adding Mixtures Formally

The Initial Idea

Expanding Homsets

If we start with a dagger compact closed category $\mathcal{C}$, we would like “expand” our homsets with extra elements that represent probabilistic mixtures of the morphisms in $\mathcal{C}$. Ideally:

- Formation of mixtures should interact well with composition - enriched category
- Our new category should contain $\mathcal{C}$ as the “pure” information
- Our new category will also be dagger compact closed
- In the absence of further information, we want to do this “freely”, introducing only the required equations
Informational Effects as Algebra

Phenomena of Interest

Probabilistic (quantified) ambiguity:

*With confidence* $p$* it’s a “river bank”, otherwise it’s a “financial bank”*

Modelled as a family of binary operations:

$$f +^p g$$

With axioms:

$$f +^p g = g +^{1-p} f$$
$$f +^p f = f$$
$$f +^p (g +^q h) = (f +^m g) +^n h$$
Unquantified ambiguity:

*It’s either a “river bank” or a “financial bank”*

Modelled as a binary operation with:

\[ f \lor g \]

Satisfying commutativity, associativity and idempotence. An affine join semilattice.
Incomplete information:

I don’t know what “logolepsy” means

Modelled as a constant:

⊥
Combinations of ambiguity and incomplete information:

I’m 90% certain “bank” means “financial bank”, otherwise I don’t know what it refers to

“financial bank” \( +p \bot \)
The algebras for describing quantitative mixtures are the algebras of the finite distribution monad $D$. They form a category of convex algebras, denoted $\text{Convex}$. The free algebra on a set $F$ is given by all formal convex combinations:

$$\sum_i p_i |f_i\rangle$$
The algebras for describing unquantified mixing are the algebras of the **non-empty finite powerset monad** $P_\omega^+$. They form a category of **affine join semilattices**, denoted $\text{AJSLat}$. The free algebra on a set $F$ is given by all non-empty finite subsets of $F$. 
The algebras for describing incomplete information are the algebras of the **lift monad** \((-\_\_\_\_\_\_)\). They form a category of **pointed sets**, denoted \(\text{Set}_\bullet\). The free algebra on a set \(F\) is given by extending \(F\) with a \(\_\_\_\_\_\_\_\) element.
The algebras for describing a combination of quantified ambiguity and incomplete information are the algebras of the subdistribution monad $S$. They form a category of subconvex algebras, denoted $\text{Subconvex}$. The free algebra on a set $F$ is given by all formal subconvex combinations:

$$ \sum_i p_i | f_i \rangle $$
The algebras for description a combination of unquantified ambiguity and incomplete information are the algebras of the finite powerset monad $P_\omega$. They form a category of join semilattices, denoted $\text{JSLat}$. The free algebra on a set $F$ is given by the set of finite subsets of $F$. 
Informational Effects as Algebras
Enrichment and the Monad Connection

Categories of Algebras

Using standard results, the Eilenberg-Moore category of a \textbf{Set} monad is:

- Complete.
- Cocomplete.
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Categories of Algebras

Using standard results, the Eilenberg-Moore category of a **Set** monad is:

- Complete.
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In fact, all the monads we’re interested in are commutative. In this case (Kock, Jacobs), the Eilenberg-Moore category:

- Is symmetric monoidal closed.
- Has universal bimorphisms for the monoidal tensor.
- Has monoidal unit given by the free algebra (\{\ast\}, !).
- Has tensors of free algebras given by free algebras on the cartesian product.
A category is:

- **Set**: enriched if its homsets are pointed sets such that:
  \[ \bot \circ f = \bot \quad \text{and} \quad f \circ \bot = \bot \]

- **AJSLat**: enriched if its homsets are affine semilattices such that:
  \[(f \vee g) \circ h = (f \circ h) \vee (g \circ h) \quad \text{and} \quad f \circ (g \vee h) = (f \circ g) \vee (f \circ h)\]

- **Convex**: enriched if its homsets are convex algebras such that:
  \[\left( \sum_i p_i f_i \right) \circ g = \sum_i p_i (f_i \circ g) \quad \text{and} \quad f \circ \left( \sum_i p_i g_i \right) = \sum_i p_i (f \circ g_i)\]
Results

For a dagger compact closed category $\mathcal{C}$ we can:

- Construct a category $\mathcal{C}_\bot$ who’s homsets contain an additional $\bot$ element.
- Construct a category $\mathcal{C}_{P^\omega}$ with homsets containing non empty subsets of $\mathcal{C}$-morphisms.
- Construct a category $\mathcal{C}_D$ with homsets containing all formal convex sums of $\mathcal{C}$-morphisms.
- Construct a category $\mathcal{C}_{P_\omega}$ with homsets containing finite subsets of $\mathcal{C}$-morphisms.
- Construct a category $\mathcal{C}_S$ with homsets containing all formal subconvex sums of $\mathcal{C}$-morphisms.
Results

In each case we can extend the composition and monoidal structure in $\mathcal{C}$ giving a category such that:

- There is an identity on objects embedding of $\mathcal{C}$ into the new category
- It is the free enrichment of $\mathcal{C}$ with respect to the appropriate algebraic structure.
- The category is dagger compact closed.
Conclusion

- We can “expand” homsets to incorporate various informational effects
- Our extended homsets contain exactly the additional elements that we need
- Even starting with categories such as $\text{Rel}$ we can form genuine probabilistic mixtures if we need them

Is it a coincidence that all these monads are commutative? Are there other information based phenomena that can be incorporated? The general process needs making explicit.
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- Our extended homsets contain exactly the additional elements that we need.
- Even starting with categories such as $\text{Rel}$ we can form genuine probabilistic mixtures if we need them.
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- Are there other information based phenomena that can be incorporated?
- The general process needs making explicit.