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# Operational meanings of orders of observables defined through quantum set theories with different conditionals

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# Classical Physics

- **Physical system** $\Leftrightarrow$ **Borel space** $(\Omega, \mathcal{F})$
- **Observables** $\Leftrightarrow$ **Real Borel functions** $X(\omega)$
- **States** $\Leftrightarrow$ **Probability measures** $P$
- $\Pr\{X \in I \| P\} = P(\{\omega \in \Omega | X(\omega) \in I\})$

# Quantum Physics

- **Physical system** $\Leftrightarrow$ **Hilbert space** $\mathcal{H}$
- **Observables** $\Leftrightarrow$ **Self-adjoint operators** $X$
- **States** $\Leftrightarrow$ **Density operators** $\rho$
- $\Pr\{X \in I \| \rho\} = \mathrm{Tr}[E^X(I)\rho]$

## Problem

- **In classical physics, the probabilities for equality and order are defined.**
- **Equality:** $\Pr\{X = Y \| P\} = P(\{\omega \in \Omega | X(\omega) = Y(\omega)\})$
- **Order:** $\Pr\{X \le Y \| P\} = P(\{\omega \in \Omega | X(\omega) \le Y(\omega)\})$
- **Problem: How should we define the probabilities for equality and order of quantum observables?** $\Pr\{X = Y \| \rho\} =?$, $\Pr\{X \le Y \| \rho\} =?$,
- **Method: Systematic use of quantum set theory.**
- **But, quantum logic has ambiguity for conditional: three candidates**
- **Conclusion: Each conditional defines a quantum set theory satisfying the ZFC transfer principle. Equality does not depend on the choice of conditional. Order depends on it, but has clear operational meaning.**

# Quantum Logic

- $\mathcal{Q}$ = **the set of projection operators on** $\mathcal{H}$**.**

  $P \leq Q \Leftrightarrow PQ = P$

  $P^{\perp} = I - P$

  $\Rightarrow \mathcal{Q}$ **is a complete orthomodular lattice.**

  $P \wedge Q = \text{wo-lim}(PQ)^n$

  $P \vee Q = (P^{\perp} \wedge Q^{\perp})^{\perp}$

# Quantum Conditionals

- **Hardegree's condition for material conditional:**

**(LB)** If $[P, Q] = 0$ then $P \rightarrow Q = P^{\perp} \vee Q$.

**(E)** $P \rightarrow Q = 1$ if and only if $P \leq Q$.

**(MP)** $P \wedge (P \rightarrow Q) \leq Q$ (modus ponens) .

**(MT)** $Q^{\perp} \wedge (P \rightarrow Q) \leq P^{\perp}$ (modus tollens).

- **There are exactly three polynomial material conditionals:**

**(S)** $P \rightarrow {}_S Q := P^{\perp} \vee (P \wedge Q)$ (Sasaki) ,

**(C)** $P \rightarrow {}_C Q := (P \vee Q)^{\perp} \vee Q$ (Contrapositive Sasaki) ,

**(R)** $P \rightarrow {}_R Q := (P \wedge Q) \vee (P^{\perp} \wedge Q) \vee (P^{\perp} \wedge Q^{\perp})$ (Relevance).

- **Note:** $P \rightarrow Q = P^{\perp} \vee Q$ does not satisfy (E).

# Characterization

- **For any $P, Q \in \mathcal{Q}$, we have the following relations.**

  (i) $P \rightarrow_S Q = \mathrm{ran}(P^\perp Q)$.

  (ii) $P \rightarrow_C Q = \mathrm{ran}(Q P^\perp)$.

  (iii) $P \rightarrow_R Q = \mathrm{ran}(P^\perp Q) \wedge \mathrm{ran}(Q P^\perp)$.

- **Biconditional is defined by**

$$P \leftrightarrow Q := (P \rightarrow Q) \wedge (Q \rightarrow P).$$

- **Biconditionals are the same:**

$$P \leftrightarrow_S Q = P \leftrightarrow_C Q = P \leftrightarrow_R Q = (P \wedge Q) \vee (P^\perp \wedge Q^\perp).$$

# Quantum Set Theory

- $V_\alpha^{(\mathcal{Q})}$ **is defined for every ordinal $\alpha$ by**

$$V_\alpha^{(\mathcal{Q})} = \{u |\ u : \mathcal{D}(u) \to \mathcal{Q}, (\exists\beta < \alpha)\ \mathcal{D}(u) \subseteq V_\beta^{(\mathcal{Q})}\},$$

**where $\mathcal{D}(u)$ is the domain of $u$.**

- **The $\mathcal{Q}$-valued universe $V^{(\mathcal{Q})}$ is defined by**

$$V^{(\mathcal{Q})} = \bigcup_{\alpha \in \mathbf{On}} V_\alpha^{(\mathcal{Q})}$$

# $\mathcal{Q}$-Valued Interpretation

- **$\mathcal{Q}$-valued ruth value $[\![\phi]\!]$ is define by the following recursion.**

  1. $[\![u = v]\!] = \bigwedge_{u' \in \mathcal{D}(u)} (u(u') \rightarrow [\![u' \in v]\!]) \wedge \bigwedge_{v' \in \mathcal{D}(v)} (v(v') \rightarrow [\![v' \in u]\!])$.
  2. $[\![u \in v]\!] = \bigvee_{v' \in \mathcal{D}(v)} (v(v') \wedge [\![u = v']\!])$.
  3. $[\![\neg\phi]\!] = [\![\phi]\!]^{\perp}$.
  4. $[\![\phi_1 \rightarrow \phi_2]\!] = [\![\phi_1]\!] \rightarrow [\![\phi_2]\!]$.
  5. $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \wedge [\![\phi_2]\!]$.
  6. $[\![(\forall x \in u)\, \phi(x)]\!] = \bigwedge_{u' \in \mathcal{D}(u)} (u(u') \rightarrow [\![\phi(u')]\!])$.
  7. $[\![(\exists x \in u)\, \phi(x)]\!] = \bigvee_{u' \in \mathcal{D}(u)} (u(u') \wedge [\![\phi(u')]\!])$.

# Embedding the Standard Universe

- **The universe $V$ of ZFC set theory is embedded by $v \mapsto \check{v}$, where $\check{v}$ is defined by**

$$\begin{aligned} \mathcal{D}(\check{v}) &= \{\check{u} |\, u \in v\}, \\ \check{v}(\check{u}) &= 1. \end{aligned}$$

**Theorem 1 (Elementary Equivalence Principle)** ***Independent of the choice of conditional, for any $\phi(x_1, \ldots, x_n)$ we have***

$$V \models \phi(u_1, \ldots, u_n) \quad \textit{if and only if} \quad [\![\phi(\check{u}_1, \ldots, \check{u}_n)]\!] = I.$$

# Commutativity

- **For any subset $\mathcal{A} \subseteq \mathcal{Q}$, the commutant of $\mathcal{A}$ is defined by**

$$\mathcal{A}^! = \{P \in \mathcal{Q} \mid [P, Q] = 0 \text{ for all } Q \in \mathcal{A}\}.$$

- **The commutator of $\mathcal{A}$ is defined by**

$$\perp\!\!\!\perp(\mathcal{A}) = \bigvee\{E \in \mathcal{A}^! \cap \mathcal{A}^{!!} \mid [P_1, P_2]E = 0 \text{ for all } P_1, P_2 \in \mathcal{A}\}.$$

- **The support $L(u)$ of $u \in V^{(\mathcal{Q})}$ is defined by recursion on the rank of $u$:**

$$L(u) = \bigcup_{x \in \mathcal{D}(u)} L(x) \cup \{u(x) \mid x \in \mathcal{D}(u)\}.$$

- **The commutator of $u_1, u_1, \ldots, u_n$ is defined by**

$$\underline{\vee}(u_1, \ldots, u_n) = \perp\!\!\!\perp(L(u_1) \cup \cdots \cup L(u_n)).$$

# Transfer Principle

**Theorem 2** ***Independent of the choice of conditional, for every formula*** $\phi(x_1, \ldots, x_n)$,

$$\textit{if } \mathrm{ZFC} \vdash \phi(x_1, \ldots, x_n) \textit{ then } \underline{\vee}(u_1, \ldots, u_n) \leq [\![\phi(u_1, \ldots, u_n)]\!].$$

# Quantum Observables as Quantum Real Numbers

- **Let $\mathrm{Q}$ be a rational numbers in $V$. The set of rational numbers in $V^{(\mathcal{Q})}$ corresponds to $\check{\mathrm{Q}}$.**

- **A real number is defined to be an upper segment of a Dedekind cut of the set of rational numbers.**

- **The predicate $\mathrm{R}(x)$ meaning "$x$ is a real number" is expressed by**

$$x \subseteq \check{\mathrm{Q}} \wedge \exists y \in \check{\mathrm{Q}}(y \in x) \wedge \exists y \in \check{\mathrm{Q}}(y \notin x)$$
$$\wedge \forall y \in \check{\mathrm{Q}}(y \in x \leftrightarrow \forall z \in \check{\mathrm{Q}}(y < z \rightarrow z \in x)).$$

- **The set $\mathrm{R}^{(\mathcal{Q})}$ of "real numbers in $V^{(\mathcal{Q})}$" is defined by**

$$\mathrm{R}^{(\mathcal{Q})} = \{u \in V^{(\mathcal{Q})} |\ \mathcal{D}(u) = \mathcal{D}(\check{\mathrm{Q}}) \text{ and } [\![\mathrm{R}(u)]\!] = 1\}.$$

**Theorem 3** ***Independent of the choice of conditional, there is a one-to-one correspondence between a real number*** $\tilde{A} = u \in \mathrm{R}^{(\mathcal{Q})}$ ***in*** $V^{(\mathcal{Q})}$ ***and a self-adjoint operator*** $A$ ***on*** $\mathcal{H}$ ***such that***

***(i)*** $E^A(\lambda) = \bigwedge_{\lambda < r \in \mathrm{Q}} u(\check{r})$ ***for every*** $\lambda \in \mathrm{R}$***,***

***(ii)*** $u(\check{r}) = E^A(r)$ ***for every*** $r \in \mathrm{Q}$***.***

# Equality for Quantum Observables

- **Independent of the choice of conditional, for any self-adjoint operators $A, B$**

$$[\![\tilde{A} = \tilde{B}]\!] = \bigwedge_{r \in Q} [\![\tilde{A} \le \check{r}]\!] \leftrightarrow [\![\tilde{B} \le \check{r}]\!] = \bigwedge_{r \in Q} E^A(r) \leftrightarrow E^B(r)$$

- **The probability of equality**

$$\mathrm{Pr}\{A = B \| \rho\} = \mathrm{Tr}[[\![\tilde{A} = \tilde{B}]\!]\rho]$$

**is independent of the choice of conditional, since so is $\leftrightarrow$.**

# Characterization of Equality

**Theorem 4** *For any observables* $\mathrm{A}$ *and* $B$ *on* $\mathcal{H}$ *and any state* $\psi \in \mathcal{H}$*, the following conditions are equivalent:*

*(i)* $\psi \Vdash \tilde{A} = \tilde{B}$*, i.e.,* $\psi \in \mathcal{R}([\![\tilde{A} = \tilde{B}]\!])$

*(ii)* $E^A(\lambda)\psi = E^B(\lambda)\psi$ *for any* $\lambda \in \mathrm{R}$.

*(iii)* $f(A)\psi = f(B)\psi$ *for every Borel function* $f$.

*(iv)* $\langle \psi, E^A(\lambda)E^B(\mu)\psi \rangle = \langle \psi, E^A(\lambda \wedge \mu)\psi \rangle$ *for any* $\lambda, \mu$.

*(v) The joint probability distribution* $\mu_\psi^{A,B}$ *exists and satisfies*

$$\mu_\psi^{A,B}(\{(a,b) \in \mathrm{R}^2 \mid a = b\}) = I.$$

# Spectral Order of Self-Adjoint Operators

- **Definition.** $X \preccurlyeq Y \Leftrightarrow E^{Y}(\lambda) \leq E^{X}(\lambda)$ **for all** $\lambda \in \mathrm{R}$.
- **Theorem (Olson, 1971). Coincides with linear order for projections and commuting self-adjoint operators.**
- **Theorem (Olson, 1971).** $0 \leq X \preccurlyeq Y \Leftrightarrow 0 \leq X^{n} \leq Y^{n}$ **for large** $n$.
- **Theorem 5** ***Independent of the choice of conditional, we have***

$$[\![\tilde{X} \leq \tilde{Y}]\!] = 1 \quad \Leftrightarrow \quad X \preccurlyeq Y$$

- **Proof: In any choice of $\to$, we have**

$$I = [\![\tilde{X} \leq \tilde{Y}]\!] = \bigwedge_{r \in Q} [\![\tilde{Y} \leq \check{r}]\!] \to [\![\tilde{X} \leq \check{r}]\!] = \bigwedge_{r \in Q} E^{Y}(r) \to E^{X}(r).$$

**Thus,** $E^{Y}(r) \to E^{X}(r) = I$ **and** $E^{Y}(r) \leq E^{X}(r)$ **by (E) for all** $r \in \mathrm{Q}$.

# Probabilistic Interpretation of the Order of Observables

- We assume $\dim(\mathcal{H}) < \infty$.
- The joint probability of obtaining the outcomes $X = x$ and $Y = y$ in the projective measurement of $Y$ immediately followed by a measurement of $X$ is given by

$$P_{\psi}^{X,Y}(x, y) = \|E^X(\{x\})E^Y(\{y\})\psi\|^2.$$

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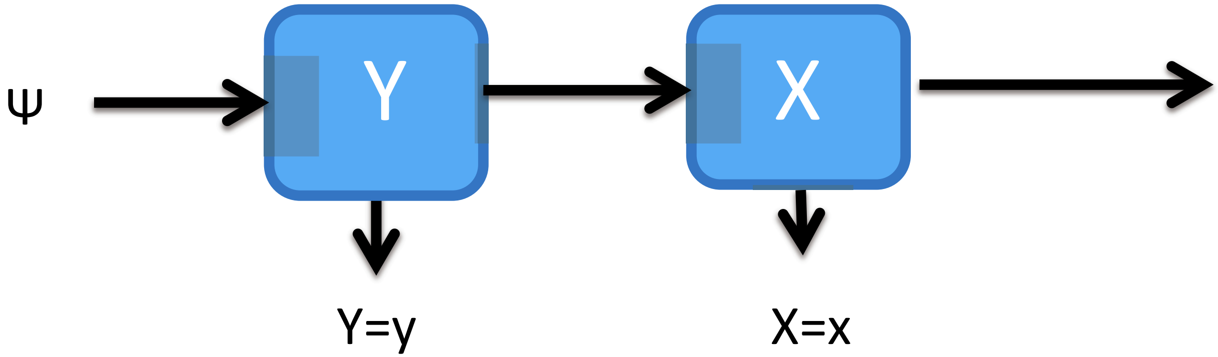


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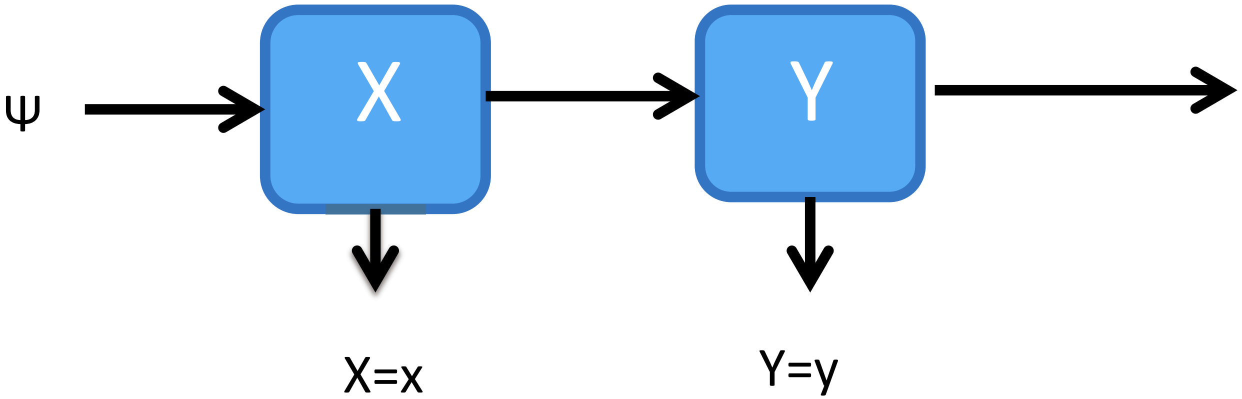

- **Theorem 6** ***For any observables $X, Y$ and a state vector $\psi$, we have the following.***

  *(i)* $\Pr\{(\tilde{X} \leq \tilde{Y})_S \| \psi\} = 1 \Leftrightarrow \sum_{(x,y):x \leq y} P_\psi^{X,Y}(x, y) = 1.$

  *(ii)* $\Pr\{(\tilde{X} \leq \tilde{Y})_C \| \psi\} = 1 \Leftrightarrow \sum_{(x,y):x \leq y} P_\psi^{Y,X}(y, x) = 1.$

  *(iii)* $\Pr\{(\tilde{X} \leq \tilde{Y})_R \| \psi\} = 1$

  $\Leftrightarrow \sum_{(x,y):x \leq y} P_\psi^{X,Y}(x, y) = 1$ *and* $\sum_{(x,y):x \leq y} P_\psi^{Y,X}(y, x) = 1.$

# Conclusion

- **In quantum mechanics, we can define the probability of equality and order relation for observables.**
- **Equality:** $\Pr\{\boldsymbol{X} = \boldsymbol{Y} \| \boldsymbol{\rho}\} = \mathrm{Tr}[\bigwedge_{r \in \mathrm{Q}} \boldsymbol{E}^{\boldsymbol{X}}(r) \leftrightarrow \boldsymbol{E}^{\boldsymbol{Y}}(r)\boldsymbol{\rho}]$
- **Order:** $\Pr\{\boldsymbol{X} \le \boldsymbol{Y} \| \boldsymbol{\rho}\} = \mathrm{Tr}[\bigwedge_{r \in \mathrm{Q}} \boldsymbol{E}^{\boldsymbol{Y}}(r) \rightarrow \boldsymbol{E}^{\boldsymbol{X}}(r)\boldsymbol{\rho}]$
- **Equality implies commutativity:** $[\![\tilde{\boldsymbol{X}} = \tilde{\boldsymbol{Y}}]\!] \le \veebar(\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}})$
- **We have**

$$\Pr\{\boldsymbol{X} = \boldsymbol{Y} \| \boldsymbol{\rho}\} = \sum_{x \in \mathrm{R}} \mathrm{Tr}[\boldsymbol{E}^{\boldsymbol{X}}(\{x\}) \wedge \boldsymbol{E}^{\boldsymbol{Y}}(\{x\})\boldsymbol{\rho}].$$

- **Order relation depends on the choice of conditional:**
- $\Pr\{(\tilde{X} \leq \tilde{Y})_S \| \psi\} = 1$: $X \leq Y$ **holds in projective** $Y$**-**$X$ **measurement (inference from past large to future small).**
- $\Pr\{(\tilde{X} \leq \tilde{Y})_C \| \psi\} = 1$: $X \leq Y$ **holds in projective** $X$**-**$Y$ **measurement (inference from past small to future large).**
- $\Pr\{(\tilde{X} \leq \tilde{Y})_R \| \psi\} = 1$: $X \leq Y$ **holds in both projective** $X$**-**$Y$ **measurement and projective** $Y$**-**$X$ **measurement (inference from both sides).**