A Royal Road to Quantum Theory
(or thereabouts)

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Recent reconstructions of (finite-dimensional) QM from simple principles all assume

- Local tomography (LT), ruling out real and quaternionic QM,
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This talk *fails* to derive f.d. QM from *simpler* principles — but gets close, with much less effort:

- No use of LT;
- Allows real, complex and quaternionic QM, plus bits of any dimension — but little else;
- Added payoff: *much easier!*

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OUTLINE:

I  Background on Jordan algebras
II  General probabilistic models
III  Conjugates and self-duality
IV  Filters and homogeneity
Let $E$ be a f.d. ordered real vector space with positive cone $E_+$ and with an inner product $\langle \cdot, \cdot \rangle$. $E$ is

- **self-dual** iff $\langle a, b \rangle \geq 0 \forall b \in E_+$ iff $a \in E_+$.
- **homogeneous** iff group of order-atomorphisms of $E$ transitive on interior of $E_+$. 

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**Background: all you need to know about Jordan algebras**

Koecher-Vingerg Theorem [1957/1961]: $E$ is HSD $\iff$ $E$ a formally real Jordan algebra with $E_+ = \{a^2 | a \in E\}$.

Jordan-von Neumann-Wigner Classification [1932]: Formally real Jordan algebras = direct sums of self-adjoint parts of $M_n(F)$, $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, M_3(\mathbb{O})$, or "spin factors" $V_n$ ("bit" with state space an $n$-ball.)
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Self-duality in QM

\( \mathcal{H} \) a complex Hilbert space, \( \dim(\mathcal{H}) = n \). Let \( E = \mathcal{L}_h(\mathcal{H}) \) with \( E_+ = \text{cone of positive operators} \). This is SD w.r.t.

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\langle a, b \rangle := \frac{1}{n} \text{Tr}(ab).
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Note that $\langle \quad \rangle = \frac{1}{n} \text{Tr}$ is a bipartite state: if

$$\psi = \frac{1}{\sqrt{n}} \sum_{x \in E} x \otimes \overline{x} \in \mathcal{H} \otimes \overline{\mathcal{H}},$$

$E$ any ONB for $\mathcal{H}$, then $\langle (a \otimes b), \psi, \psi \rangle = \frac{1}{n} \text{Tr}(ab)$. 

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\( E \) any ONB for \( \mathcal{H} \), then \( \langle (a \otimes \bar{b}), \Psi, \Psi \rangle = \frac{1}{n} \text{Tr}(ab) \).

So \( \Psi \) perfectly, and uniformly correlates every ONB of \( \mathcal{H} \) with its counterpart in \( \overline{\mathcal{H}} \): 

\[ |\langle \Psi, x \otimes \bar{y} \rangle|^2 = \frac{1}{n} \text{ if } x = y, 0 \text{ if } x \perp y. \]

\( \Psi \) is uniquely defined by this feature.
Probabilistic models

A **test space**: a collection $\mathcal{M} = \{E, F, \ldots\}$ of (outcome-sets of) possible measurements, experiments, *tests*, etc.

Let $X := \bigcup \mathcal{M}$. A **probability weight** on $\mathcal{M}$:

$$\alpha : X \to [0, 1] \text{ with } \sum_{x \in E} \alpha(x) = 1 \ \forall E \in \mathcal{M}.$$ 

A **probabilistic model**: a pair $A = (\mathcal{M}, \Omega)$,
• $\mathcal{M}$ a test space,
• $\Omega$ a convex set of probability weights on $\mathcal{M}$, the *states* of $A$.

*Notation*: $\mathcal{M}(A)$, $X(A)$ and $\Omega(A)$ ...
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**Standing assumption**: $\Omega(A)$ finite-dimensional.
Two important examples

Simple classical model: \( A = (\{ E \}, \Delta(E)) \) — one test, all probability weights.

Simple quantum model: For a (f.d.) Hilbert space \( \mathcal{H} \), let

- \( \mathcal{M}(\mathcal{H}) = \) set of ONBs for \( \mathcal{H} \);
- \( \Omega(\mathcal{H}) = \) all probability weights states of the form

\[ \alpha(x) = \langle Wx, x \rangle, \]

\( W \) a density operator on \( \mathcal{H} \). (= all prob. weights, if \( \dim \mathcal{H} > 2 \).)
Two-bit examples

The square bit $B$ and diamond bit $B'$ have the same test space:

$$\mathcal{M}(B) = \mathcal{M}(B') = \{\{x, x'\}, \{y, y'\}\}$$

but different state spaces:

$$\Omega(A) = \text{all prob weights on } \mathcal{M}(A)$$

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Some properties of probabilistic models

A probabilistic model $A$ is

- **uniform** iff all tests $E \in \mathcal{M}(A)$ have a common size, say $|E| = n$ (the *rank* of $A$)
- **sharp** iff $\forall x \in X(A) \exists! \delta_x \in \Omega(A)$ with $\delta_x(x) = 1$;
- **spectral** iff sharp and, $\forall \alpha \in \Omega(A)$, $\exists E \in \mathcal{M}(A)$ with

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\alpha = \sum_{x \in E} \alpha(x) \delta_x.
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$$\alpha = \sum_{x \in E} \alpha(x) \delta_x.$$

Square bit $\rightarrow$ uniform, but not sharp.
Diamond bit $\rightarrow$ uniform and sharp, but not spectral.
Classical and quantum models $\rightarrow$ uniform, sharp, spectral.
The spaces $\mathbf{V}(A)$ and $\mathbf{E}(A)$

$\mathbf{V}(A) = \text{span of } \Omega(A) \text{ in } \mathbb{R}^{X(A)}, \text{ with positive cone}$

$$\mathbf{V}(A)_+ := \{ t\alpha \mid \alpha \in \Omega, \ t \geq 0 \}$$
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Effects are elements $a \in \mathbf{V}(A)^*$ with $0 \leq a(\alpha) \leq 1 \ \forall \alpha \in \Omega(A)$.

Example: $\hat{x}(\alpha) = \alpha(x)$ for $x \in X(A)$. Note: $\forall E \in \mathcal{M}(A)$,

$$\sum_{x \in E} \hat{x} =: u_A, \ u_A(\alpha) = 1 \ \forall \alpha \in \Omega(A).$$
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It’s also useful to define $E(A) = V(A)^*$, but ordered by

$$E(A)_+ := \left\{ \sum_{i=1}^{k} t_i \hat{x}_i \mid x_i \in X(A), \ t_i \geq 0 \right\}$$
Joint States

A (non-signaling) **joint state** on $A$ and $B$ is a mapping

$$\omega : X(A) \times X(B) \to [0, 1]$$

with

(a) $(E, F) \in \mathcal{M}(A) \times \mathcal{M}(B) \implies \sum_{(x,y) \in E \times F} \omega(x, y) = 1$;

(b) $x \in X(A), y \in X(B) \implies \omega(x \cdot ) \in \mathcal{V}_+(B)$ and $\omega(\cdot y) \in \mathcal{V}_+(A)$

Condition (b) ensures that $\omega \in \Omega(AB)$ has well-defined **marginal and conditional states**:

$$\omega_1(x) := \sum_{y \in F} \omega(\cdot, y) \in \Omega(A) \quad \text{and} \quad \omega_{2|x}(y) := \frac{\omega(x, y)}{\omega_1(x)} \in \Omega(B);$$

similarly for $\omega_2(y), \omega_{1|y}$.
Joint States

Marginal and conditional states are related by a **Law of total probability**: \( \forall E \in \mathcal{M}(A), F \in \mathcal{M}(B), \)

\[
\omega_2 = \sum_{x \in E} \omega_1(x)\omega_2|_x \quad \text{and} \quad \omega_1 = \sum_{y \in F} \omega_2(y)\omega_1|_y
\]

**Lemma 0:** *Every joint state extends to a unique positive linear mapping*

\[ \hat{\omega} : E(A) \rightarrow V(B), \]

*such that \( \hat{\omega}(x)(y) = \omega(x, y) \forall x \in X(A), y \in X(B). \)*
Conjugates

Let $A$ be uniform, with rank $n$. A **conjugate** for $A$: a triple $(\overline{A}, \gamma_A, \eta_A)$, $\gamma_A : A \simeq \overline{A}$ an isomorphism and $\eta_A$ is a joint state on $A$ and $\overline{A}$ such that

(a) $\eta(x, \gamma_A(y)) = \eta(y, \gamma_A(x))$ and

(b) $\eta_A(x, \gamma_A(x)) = \frac{1}{n} \ \forall x \in X(A)$.

**Notation:** $\gamma_A(x) =: \overline{x}$.

Note that $(\eta_A)_{1|\overline{x}}(x) = 1$. Thus, $A$ sharp $\Rightarrow \eta_A$ uniquely defined (by $\eta_A(x, \overline{y}) = \frac{1}{n} \delta_y(x)$) $\Rightarrow \eta_A$ is symmetric.
Lemma 1: Let $A$ be sharp, spectral, and have a conjugate. Then

\[ \langle a, b \rangle := \eta_A(a, b) \]

is a self-dualizing inner product on $E(A)$.

Proof: Exercise!
**Lemma 1:** Let $A$ be sharp, spectral, and have a conjugate. Then

$$\langle a, b \rangle := \eta_A(a, \bar{b})$$

is a self-dualizing inner product on $E(A)$.

**Proof:** Exercise!

**Hints:** $\langle \ , \ \rangle$ bilinear and symmetric by Lemma 0 and sharpness. By spectrality, $\hat{\eta}$ takes $E(A)_+$ onto $V(A)_+$, so, is an order-isomorphism. Spectrality now also implies every $a \in E(A)$ has a decomposition $a = \sum_{x \in E} t_x x$ for some $E \in \mathcal{M}(A)$ and coefficients $t_x \in \mathbb{R}$. Hence,

$$\langle a, a \rangle = \sum_{x, y \in E \times E} t_x t_y \eta_A(x, \bar{y}) = \frac{1}{n} \sum_{x \in E} t_x^2 \geq 0,$$

with equality only where $a = 0$. So $\langle \ , \ \rangle$ is positive-definite. That it’s self-dualizing follows easily from $\hat{\eta}$’s being an order-isomorphism. □
Two Corollaries

Let $A$ satisfy the assumptions of Lemma 1. Then

**Corollary 1 (Spectral Uniqueness Theorem):** Every $a \in E(A)$ has a unique expansion $a = \sum_i t_i e_i$ with $e_i$ sharply distinguishable effects and $t_i$ distinct.

This a gives us a functional calculus: with $a = \sum_i t_i e_i$ as above, define

$$f(a) = \sum_i f(t_i) e_i.$$ 

**Corollary 2:** If $\mathcal{M}(A)$ has rank two then the state space $\Omega(A)$ is a euclidean ball (hence, $E(A)$ is a spin factor).
Processes

A **process** from $A$ to $B$ is represented by a positive linear mapping

$$\tau : \mathbf{V}(A) \to \mathbf{V}(B) \text{ with } u_B(\tau(\alpha)) \leq 1 \forall \alpha \in \Omega(A).$$

Can think of $p = u_B(\tau(\alpha))$ as probability for the process to “fail” on input state $\alpha$.

(Not every such mapping need count as a processes!)

$\tau$ is **reversible** iff $\exists$ a process $\tau'$ such that $\tau' \circ \tau = p\text{id}$: with probability $p$, $\tau'$ reverses $\tau$.

This implies $\tau$ is invertible with $\tau^{-1}$ positive, i.e., $\tau$ is an order-automorphism.
Filters and Homogeneity

A **filter** for \( E \in \mathcal{M}(A) \): a process \( \Phi : \mathcal{V}(A) \rightarrow \mathcal{V}(A) \) such that
\[
\forall x \in E \; \exists t_x \geq 0 \text{ with } \Phi(\alpha)(x) = t_x \alpha(x)
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for all \( \alpha \in \Omega(A) \).
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**Example:** For a density operator on $\mathcal{H}$, $\Phi : a \mapsto W^{1/2}aW^{1/2}$ is a filter for any eigenbasis of $W$, reversible iff $W$ is nonsingular.
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Appealing to the KV Theorem,

**Theorem 1:** Let $A$ satisfy hypotheses of Lemma 1. Then $\text{TAE}$:

(a) $A$ has arbitrary reversible filters
(b) $V(A)$ is homogeneous
(c) $E(A)$ is a formally real Jordan algebra.

One can also show that then $X(A)$ is the set of all minimal idempotents in $E$, and $\mathcal{M}(A)$ is the set of Jordan frames, i.e., $A$ is a *Jordan model* (see arXiv: 1206.2897).
Why spectrality?

A joint state $\omega \in \Omega(AB)$ **correlating** iff $\exists E \in \mathcal{M}(A), F \in \mathcal{M}(B),$ and partial bijection $f \subseteq E \times F$ such that

$$\omega(x, y) > 0 \iff (x, y) \in f.$$ 

**Lemma 2:** A sharp and $\omega \in \Omega(AB)$, correlating $\Rightarrow \omega_1$ spectral.

*Proof:* With $f \subseteq E \times F$ as above, $\omega_1|_{f(x)}(x) = 1$, so $\omega_1|_x(f(x)) = \delta_x$. By LOTP, $\alpha = \sum_{x \in \text{dom}(f)} \omega_2(f(x))\delta_x$. □

**Correlation Postulate:** Every state is the marginal of a correlating joint state.

So: CP implies spectrality. (Note affinity with the “purification postulate” of Chiribella et al.)
Memory and Correlation

Can the CP itself be further motivated?

Suppose the outcome of a test $E \in \mathcal{M}(A)$ is recorded in the state of an ancilla $B$. Then $A$ and $B$ must be in a joint state $\omega$ such that the conditional states $\omega_{2|x} := \beta_x$, $x \in E$, are sharply distinguishable, say by $F \in \mathcal{M}(B)$. Then $\omega$ correlates $E$ with $F$. If the measurement of $E$ doesn’t disturb $\alpha$, then $\alpha = \omega_1$.

So we might adopt

**Non-Disturbance Principle:** For every state, there is a test that can be recorded in that state without disturbance.
Conclusion:

Four conditions characterize probabilistic models associated with formally real Jordan algebras:

(1) $A$ is sharp,
(2) $A$ has a conjugate,
(3) $A$ satisfies the CP
(4) $A$ has arbitrary reversible filters

Condition (4) is needed only for homogeneity. Conditions (1)-(3) already yield a rich structure (Corollaries 1, 2).

Questions:

- Can one prove Theorem 1 without using the KV theorem?
- Can Lemma 1 help simplify earlier reconstruction results?
- Monoidal categories of probabilistic models having well-behaved conjugates are automatically dagger-compact, with $\eta_A$ as “cup”. In such a category, is spectrality automatic?
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Thanks!