A Royal Road to Quantum Theory (or thereabouts)

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Goal and Outline

Recent reconstructions of (finite-dimensional) QM from simple principles [1] all assume

- Local tomography (LT), ruling out real and quaternionic QM,
- Systems are determined by their "information capacity" (so, only one type of bit).

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This talk *fails* to derive f.d. QM from *simpler* principles — but gets close, with much less effort:

- No use of LT;
- Allows real, complex and quaternionic QM, plus bits of any dimension – but little else;
- Added payoff: *much easier!*

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OUTLINE:

I *Background on Jordan algebras*

II *General probabilistic models*

III *Conjugates and self-duality*

IV *Filters and homogeneity*

Background: all you need to know about Jordan algebras

Let $\mathbf{E}$ be a f.d. ordered real vector space with positive cone $\mathbf{E}_+$ and with an inner product $\langle \, , \, \rangle$. $\mathbf{E}$ is

- *self-dual* iff $\langle a, b \rangle \geq 0 \; \forall b \in \mathbf{E}_+$ iff $a \in \mathbf{E}_+$.
- *homogeneous* iff group of order-atomorphisms of $\mathbf{E}$ transitive on interior of $\mathbf{E}_+$.

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Self-duality in QM

$\mathcal{H}$ a complex Hilbert space, $\dim(\mathcal{H}) = n$. Let $\mathbf{E} = \mathcal{L}_h(\mathcal{H})$ with $\mathbf{E}_+$ = cone of positive operators. This is SD w.r.t.

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Note that $\langle \quad \rangle = \frac{1}{n}\mathrm{Tr}$ is a *bipartite state*: if

$$\mathbf{\Psi} = \frac{1}{\sqrt{n}} \sum_{x \in E} x \otimes \overline{x} \in \mathcal{H} \otimes \overline{\mathcal{H}},$$

E any ONB for $\mathcal{H}$, then $\big\langle (a \otimes \overline{b}), \Psi, \Psi \big\rangle \; = \; \frac{1}{n}\mathrm{Tr}(ab)$.

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So $\mathbf{\Psi}$ perfectly, and uniformly correlates every ONB of $\mathcal{H}$ with its counterpart in $\overline{\mathcal{H}}$: $|\langle \mathbf{\Psi}, x \otimes \overline{y} \rangle|^2 = \frac{1}{n}$ if $x = y$, 0 if $x \perp y$. $\mathbf{\Psi}$ *is uniquely defined by this feature.*

Probabilistic models

A **test space**: a collection $\mathcal{M} = \{E, F, ...\}$ of (outcome-sets of) possible measurements, experiments, *tests*, etc.

Let $X := \bigcup \mathcal{M}$. A **probability weight** on $\mathcal{M}$:

$$\alpha : X \to [0, 1] \text{ with } \sum_{x \in E} \alpha(x) = 1 \ \forall E \in \mathcal{M}.$$

A **probabilistic model**: a pair $A = (\mathcal{M}, \Omega)$,

- $\mathcal{M}$ a test space,
- Ω a convex set of probability weights on $\mathcal{M}$, the *states* of A.

Notation: $\mathcal{M}(A)$, $X(A)$ and $\Omega(A)$ ...

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Standing assumption: $\Omega(A)$ finite-dimensional.

Two important examples

Simple **classical model**: $A = (\{E\}, \Delta(E))$ — one test, all probability weights.

Simple **quantum model**: For a (f.d.) Hilbert space $\mathcal{H}$, let

- $\mathcal{M}(\mathcal{H})$ = set of ONBs for $\mathcal{H}$;
- $\Omega(\mathcal{H})$ = all probability weights states of the form

$$\alpha(x) = \langle Wx, x \rangle,$$

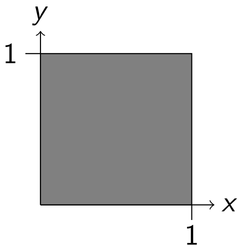
W a density operator on $\mathcal{H}$. (= all prob. weights, if $\dim\mathcal{H} > 2$.)

Two-bit examples

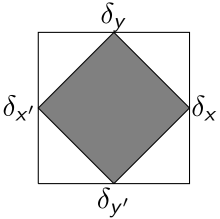
The **square bit** B and **diamond bit** B' have the same test space:

$$\mathcal{M}(B) = \mathcal{M}(B') = \{\{x, x'\}, \{y, y'\}\}$$

but different state spaces:



$\Omega(A) =$ all prob weights on $\mathcal{M}(A)$



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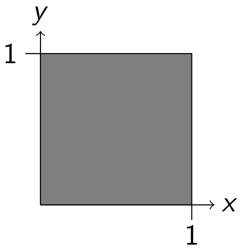

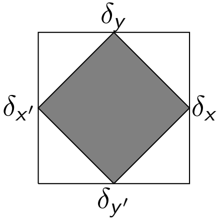

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Some properties of probabilistic models

A probabilistic model A is

- **uniform** iff all tests $E \in \mathcal{M}(A)$ have a common size, say $|E| = n$ (the *rank* of A)
- **sharp** iff $\forall x \in X(A)\ \exists! \delta_x \in \Omega(A)$ with $\delta_x(x) = 1$;
- **spectral** iff sharp and, $\forall \boldsymbol{\alpha} \in \Omega(A)$, $\exists E \in \mathcal{M}(A)$ with

$$\boldsymbol{\alpha} = \sum_{x \in E} \alpha(x) \delta_x.$$

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Square bit $\rightarrow$ uniform, but not sharp.
Diamond bit $\rightarrow$ uniform and sharp, but not spectral.
Classical and quantum models $\rightarrow$ uniform, sharp, spectral.

The spaces $\mathbf{V}(A)$ and $\mathbf{E}(A)$

$\mathbf{V}(A)$ = span of $\Omega(A)$ in $\mathbb{R}^{X(A)}$, with positive cone

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Effects are elements $a \in \mathbf{V}(A)^*$ with $0 \leq a(\alpha) \leq 1\ \forall \boldsymbol{\alpha} \in \Omega(A)$.
Example: $\widehat{x}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}(x)$ for $x \in X(A)$. Note: $\forall E \in \mathcal{M}(A)$, $\sum_{x \in E} \widehat{x} =: u_A$, $u_A(\boldsymbol{\alpha}) = 1$ forall $\boldsymbol{\alpha} \in \Omega(A)$.

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It's also useful to define $\mathbf{E}(A) = \mathbf{V}(A)^*$, but ordered by

$$\mathbf{E}(A)_+ := \left\{ \sum_{i=1}^{k} t_i \widehat{x}_i \,\middle|\, x_i \in X(A),\ t_i \geq 0 \right\}$$

Joint States

A (non-signaling) **joint state** on A and B is a mapping

$$\omega : X(A) \times X(B) \to [0,1]$$

with

(a) $(E,F) \in \mathcal{M}(A) \times \mathcal{M}(B) \implies \sum_{(x,y)\in E\times F} \omega(x,y) = 1$;

(b) $x \in X(A)$, $y \in X(B) \implies$

$$\omega(x\ \cdot\ ) \in \mathbf{V}_+(B) \text{ and } \omega(\ \cdot\ y) \in \mathbf{V}_+(A)$$

Condition (b) ensures that $\omega \in \Omega(AB)$ has well-defined **marginal and conditional states**:

$$\omega_1(x) := \sum_{y\in F} \omega(\cdot, y) \in \Omega(A) \quad \text{and} \quad \omega_{2|x}(y) := \frac{\omega(x,y)}{\omega_1(x)} \in \Omega(B);$$

similarly for $\omega_2(y), \omega_{1|y}$.

Joint States

Marginal and conditional states are related by a **Law of total probability:** $\forall\ E \in \mathcal{M}(A),\ F \in \mathcal{M}(B)$,

$$\omega_2 = \sum_{x \in E} \omega_1(x)\omega_{2|x} \quad \text{and} \quad \omega_1 = \sum_{y \in F} \omega_2(y)\omega_{1|y}$$

Lemma 0: *Every joint state extends to a unique positive linear mapping*

$$\widehat{\omega} : \mathbf{E}(A) \to \mathbf{V}(B),$$

such that $\widehat{\omega}(x)(y) = \omega(x, y) \quad \forall x \in X(A),\ y \in X(B).$

Conjugates

Let A be uniform, with rank n. A **conjugate** for A: a triple $(\overline{A}, \gamma_A, \eta_A)$, $\gamma_A : A \simeq \overline{A}$ an isomorphism and η_A is a joint state on A and $\overline{A}$ such that

(a) $\eta(x, \gamma_A(y)) = \eta(y, \gamma_A(x))$ and

(b) $\eta_A(x, \gamma_A(x)) = \frac{1}{n}$ $\forall x \in X(A)$.

Notation: $\gamma_A(x) =: \overline{x}$.

Note that $(\eta_A)_{1|\overline{x}}(x) = 1$. Thus, A sharp $\Rightarrow$ η_A uniquely defined (by $\eta_A(x, \overline{y}) = \frac{1}{n}\delta_y(x)$) $\Rightarrow$ η_A is symmetric.

Lemma 1: *Let A be sharp, spectral, and have a conjugate. Then*

$$\langle a, b\rangle := \eta_A(a, \overline{b})$$

is a self-dualizing inner product on $\mathbf{E}(A)$.

Proof: Exercise!

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Proof: Exercise!
Hints: $\langle \ , \ \rangle$ bilinear and symmetric by Lemma 0 and sharpness. By spectrality, $\widehat{\eta}$ takes $\mathbf{E}(A)_+$ *onto* $\mathbf{V}(A)_+$, so, is an order-isomorphism. Spectrality now also implies every $a \in \mathbf{E}(A)$ has a decomposition $a = \sum_{x \in E} t_x x$ for some $E \in \mathcal{M}(A)$ and coefficients $t_x \in \mathbb{R}$. Hence,

$$\langle a, a \rangle = \sum_{x,y \in E \times E} t_x t_y \eta_A(x, \overline{y}) = \frac{1}{n} \sum_{x \in E} {t_x}^2 \geq 0,$$

with equality only where $a = 0$. So $\langle \ , \ \rangle$ is positive-definite. That it's self-dualizing follows easily from $\widehat{\eta}$'s being an order-isomorphism. □

Two Corollaries

Let A satisfy the assumptions of Lemma 1. Then

Corollary 1 (Spectral Uniqueness Theorem): *Every* $a \in \mathbf{E}(A)$ *has a* unique *expansion* $a = \sum_i t_i e_i$ *with* e_i *sharply distinguishable effects and* t_i *distinct.*

This a gives us a functional calculus: with $a = \sum_i t_i e_i$ as above, define

$$f(a) = \sum_i f(t_i) e_i.$$

Corollary 2: *If* $\mathcal{M}(A)$ *has rank two then the state space* $\Omega(A)$ *is a euclidean ball (hence,* $\mathbf{E}(A)$ *is a spin factor).*

Processes

A **process** from A to B is represented by a positive linear mapping

$$\tau : \mathbf{V}(A) \to \mathbf{V}(B) \ \text{ with } \ u_B(\tau(\boldsymbol{\alpha})) \leq 1 \ \forall \boldsymbol{\alpha} \in \Omega(A).$$

Can think of $p = u_B(\tau(\boldsymbol{\alpha}))$ as probability for the process to "fail" on input state $\boldsymbol{\alpha}$.

(Not every such mapping need count as a processes!)

τ is **reversible** iff $\exists$ a process τ' such that $\tau' \circ \tau = p\text{id}$: with probability p, τ' reverses τ.

This implies τ is invertible with τ^{-1} positive, i.e., τ is an order-automorphism.

Filters and Homogeneity

A **filter** for $E \in \mathcal{M}(A)$: a process $\Phi : \mathbf{V}(A) \to \mathbf{V}(A)$ such that $\forall x \in E \quad \exists t_x \geq 0$ with

$$\Phi(\boldsymbol{\alpha})(x) = t_x \boldsymbol{\alpha}(x)$$

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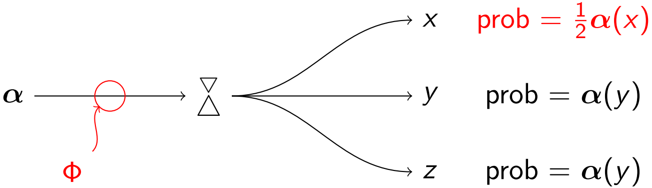

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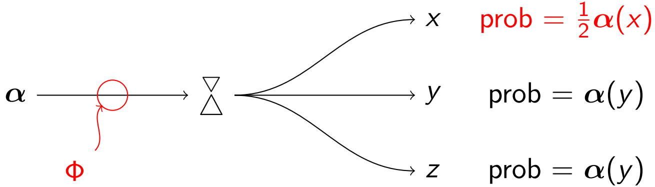

Example: For W a density operator on $\mathcal{H}$, $\Phi : a \mapsto W^{1/2} a W^{1/2}$ is a filter for any eigenbasis of W, reversible iff W is nonsingular.

Appealing to the KV Theorem,

Theorem 1: *Let A satisfy hypotheses of Lemma 1. Then TAE:*
(a) A has arbitrary reversible filters
(b) $\mathbf{V}(A)$ is homogeneous
(c) $\mathbf{E}(A)$ is a formally real Jordan algebra.

One can also show that then $X(A)$ is the set of all minimal idempotents in $\mathbf{E}$, and $\mathcal{M}(A)$ is the set of Jordan frames, i.e., A is a *Jordan model* (see arXiv: 1206.2897).

Why spectrality?

A joint state $\omega \in \Omega(AB)$ **correlating** iff $\exists E \in \mathcal{M}(A)$, $F \in \mathcal{M}(B)$, and partial bijection $f \subseteq E \times F$ such that

$$\omega(x, y) > 0 \;\Leftrightarrow\; (x, y) \in f.$$

Lemma 2: *A sharp and $\omega \in \Omega(AB)$, correlating $\Rightarrow \omega_1$ spectral.*

Proof: With $f \subseteq E \times F$ as above, $\omega_{1|f(x)}(x) = 1$, so $\omega_{1|x}(f(x)) = \delta_x$. By LOTP, $\boldsymbol{\alpha} = \sum_{x \in \mathrm{dom}(f)} \omega_2(f(x))\delta_x$. $\square$

Correlation Postulate: Every state is the marginal of a correlating joint state.

So: CP implies spectrality. (Note affinity with the "purification postulate" of Chiribella et al.)

Memory and Correlation

Can the CP itself be further motivated?

Suppose the outcome of a test $E \in \mathcal{M}(A)$ is recorded in in the state of an ancilla B. Then A and B must be in a joint state ω such that the conditional states $\omega_{2|x} := \beta_x$, $x \in E$, are sharply distinguishable, say by $F \in \mathcal{M}(B)$. Then ω correlates E with F. If the measurement of E doesn't disturb $\boldsymbol{\alpha}$, then $\boldsymbol{\alpha} = \omega_1$.

So we might adopt

Non-Disturbance Principle: For every state, there is a test that can be recorded in that state without disturbance.

Conclusion:

Four conditions characterize probabilistic models associated with formally real Jordan algebras:

(1) A is sharp,
(2) A has a conjugate,
(3) A satisfies the CP
(4) A has arbitrary reversible filters

Condition (4) is needed only for homogeneity. Conditions (1)-(3) already yield a rich structure (Corollaries 1, 2).

Questions:

- Can one prove Theorem 1 without using the KV theorem?
- Can Lemma 1 help simplify earlier reconstruction results?
- Monoidal categories of probabilistic models having well-behaved conjugates are automatically dagger-compact, with η_A as "cup". In such a category, is spectrality automatic?

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Thanks!