Quantifying Contextuality
via linear programming

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Quantum Physics & Logic
University of Strathclyde, Glasgow, 8th June 2016
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▶ unified framework for non-locality and contextuality in general measurement scenarios

Why?

▶ Comparing degree of contextuality of empirical models

▶ . . . and across different scenarios

▶ Contextuality as a resource

▶ There may be more than one useful measure
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- qualitative hierarchy of contextuality for empirical models

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Overview

We introduce the **contextual fraction** (generalising the idea of non-local fraction).

It satisfies a number of desirable properties:
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- Generality, i.e. applicable to any measurement scenario
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- Computable, using linear programming
- Precise relationship to **violations of Bell inequalities**
Contextuality
Empirical data

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>(0, 0)</td>
<td>(0, 1)</td>
<td>(1, 0)</td>
</tr>
<tr>
<td>a₁</td>
<td>b₁</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a₁</td>
<td>b₂</td>
<td>3/8</td>
<td>1/8</td>
<td>1/8</td>
</tr>
<tr>
<td>a₂</td>
<td>b₁</td>
<td>3/8</td>
<td>1/8</td>
<td>1/8</td>
</tr>
<tr>
<td>a₂</td>
<td>b₂</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
</tr>
</tbody>
</table>

{oₐ ∈ {0, 1}}

{oₐ ∈ {0, 1}}

{mₐ ∈ {a₁, a₂}}

{mₐ ∈ {a₁, a₂}}

{p}
Abramsky–Brandenburger framework

Measurement scenario $\langle X, M, O \rangle$:

- $X$ is a finite set of measurements or variables
- $O$ is a finite set of outcomes or values
- $M$ is a cover of $X$, indicating **joint measurability** (contexts)

Example: (2,2,2) Bell scenario

- The set of variables is $X = \{a_1, a_2, b_1, b_2\}$.
- The outcomes are $O = \{0, 1\}$.
- The measurement contexts are:
  - $\{a_1, b_1\}$
  - $\{a_1, b_2\}$
  - $\{a_2, b_1\}$
  - $\{a_2, b_2\}$

A joint outcome or event in a context $C$ is $s \in O^C$, e.g. $s = [a_1 \mapsto 0, b_1 \mapsto 1]$. (These correspond to the cells of our probability tables.)
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Another example: 18-vector Kochen–Specker

- A set of 18 variables, $X = \{A, \ldots, O\}$
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- A set of 18 variables, $X = \{A, \ldots, O\}$
- A set of outcomes $O = \{0, 1\}$
- A measurement cover $\mathcal{M} = \{C_1, \ldots, C_9\}$, whose contexts $C_i$ correspond to the columns in the following table:

<table>
<thead>
<tr>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$U_3$</th>
<th>$U_4$</th>
<th>$U_5$</th>
<th>$U_6$</th>
<th>$U_7$</th>
<th>$U_8$</th>
<th>$U_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>H</td>
<td>H</td>
<td>B</td>
<td>I</td>
<td>P</td>
<td>P</td>
<td>Q</td>
</tr>
<tr>
<td>B</td>
<td>E</td>
<td>I</td>
<td>K</td>
<td>E</td>
<td>K</td>
<td>Q</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>C</td>
<td>F</td>
<td>C</td>
<td>G</td>
<td>M</td>
<td>N</td>
<td>D</td>
<td>F</td>
<td>M</td>
</tr>
<tr>
<td>D</td>
<td>G</td>
<td>J</td>
<td>L</td>
<td>N</td>
<td>O</td>
<td>J</td>
<td>L</td>
<td>O</td>
</tr>
</tbody>
</table>
Empirical Models

Fix a measurement scenario $\langle X, M, O \rangle$. 

Compatibility condition: these distributions "agree on overlaps", i.e.

$$e_C |_{C \cap C'} = e_{C'} |_{C \cap C'}.$$ 

where marginalisation of distributions: if $D \subseteq C$, $d \in \text{Prob}(O_C)$,

$$d |_{D}(s) := \sum_{t \in O_C, t |_{D} = s} d(t).$$
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Fix a measurement scenario $\langle X, M, O \rangle$.

**Empirical model**: family $\{e_C\}_{C \in M}$ where $e_C \in \text{Prob}(O^C)$ for $C \in M$.

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For multipartite scenarios, compatibility = the **no-signalling** principle.
Contextuality

A (compatible) empirical model is non-contextual if there exists a global distribution $d \in \text{Prob}(O^X)$ (on the joint assignments of outcomes to all measurements) that marginalises to all the $e_C$:

$$\exists d \in \text{Prob}(O^X) \cdot \forall C \in \mathcal{M} \cdot d|_C = e_C.$$
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That is, we can glue all the local information together into a global consistent description from which the local information can be recovered.
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**Contextuality:**
family of data which is **locally consistent** but **globally inconsistent**.
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**Contextuality:**
family of data which is **locally consistent** but **globally inconsistent**.

The import of results such as Bell’s and Bell–Kochen–Specker’s theorems is that there are empirical models arising from quantum mechanics that are contextual.
Strong Contextuality: 

no event can be extended to a global assignment.
Strong contextuality

Strong Contextuality: *no* event can be extended to a global assignment.

E.g. K–S models, GHZ, the PR box:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(0, 0)</th>
<th>(0, 1)</th>
<th>(1, 0)</th>
<th>(1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$b_1$</td>
<td>✓</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$b_2$</td>
<td>✓</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$b_1$</td>
<td>✓</td>
<td>×</td>
<td>×</td>
<td>✓</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$b_2$</td>
<td>×</td>
<td>✓</td>
<td>✓</td>
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</tbody>
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The contextual fraction
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Non-contextuality: global distribution $d \in \text{Prob}(O^X)$ such that:

$$\forall C \in M. \ d|_C = e_C.$$
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Which fraction of a model admits a non-contextual explanation?
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Which fraction of a model admits a non-contextual explanation?

Consider **subdistributions** $c \in \text{SubProb}(O^X)$ such that:

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Non-contextual fraction: maximum weight of such a subdistribution.
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Equivalently, maximum weight \( \lambda \) over all convex decompositions

\[
e = \lambda e^{NC} + (1 - \lambda) e'
\]

where \( e^{NC} \) is a non-contextual model.
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\[
\text{NCF}(e) = \lambda \quad \text{CF}(e) = 1 - \lambda
\]
Computing the contextual fraction
Contextuality as a linear system

For a measurement scenario $\langle X, M, O \rangle$, the incidence matrix $M$ has

- $m$ rows indexed by $\langle C, s \rangle$, $C \in M$, $s \in O^C$
- $n$ columns indexed by global assignments $g \in O^X$

$$M[\langle C, s \rangle, g] := \begin{cases} 1 & \text{if } g|_C = s \\ 0 & \text{otherwise} \end{cases}.$$
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A probability distribution on global assignments \( O^X \) is given by a vector \( d \in \mathbb{R}^n \). The corresponding NC model is given by \( M \cdot d \).
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A model \( e \) is non-contextual if and only if there is \( d \in \mathbb{R}^n \) solving:

\[
M d = v^e \quad \text{with} \quad d \geq 0.
\]
Checking contextuality of $e$ corresponds to solving

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\text{Find } \quad d \in \mathbb{R}^n \\
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\end{align*}
\]

Computing the non-contextual fraction corresponds to solving the following linear program:

\[
\begin{align*}
\text{Find} \quad & \mathbf{c} \in \mathbb{R}^n \\
\text{maximising} \quad & 1 \cdot \mathbf{c} \\
\text{subject to} \quad & \mathbf{M} \mathbf{c} \leq \mathbf{v}^e \\
\text{and} \quad & \mathbf{c} \geq 0
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Violations of Bell inequalities
Generalised Bell inequalities

An inequality for a scenario \( \langle X, \mathcal{M}, O \rangle \) is given by:

- a set of coefficients \( \alpha = \{ \alpha(C, s) \}_{C \in \mathcal{M}, s \in \mathcal{E}(C)} \)
- a bound \( R \).

For a model \( e \), the inequality reads as

\[
B_\alpha(e) \leq R,
\]

where

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B_\alpha(e) := \sum_{C \in \mathcal{M}, s \in \mathcal{E}(C)} \alpha(C, s) e(C, s).
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Wlog we can take \( R \) non-negative (in fact, we can take \( R = 0 \)).

It is called a Bell inequality if it is satisfied by every NC model. If it is saturated by some NC model, the Bell inequality is said to be tight.

S. Abramsky, R. S. Barbosa, & S. Mansfield
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Violation of a Bell inequality

A Bell inequality establishes a bound for the value of $B_\alpha(e)$ amongst NC models.
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For a general (no-signalling) model $e$, the quantity is limited only by

$$\|\alpha\| := \sum_{C \in \mathcal{M}} \max \{ \alpha(C, s) \mid s \in \mathcal{E}(C) \}$$
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A Bell inequality establishes a bound for the value of $B_\alpha(e)$ amongst NC models.

For a general (no-signalling) model $e$, the quantity is limited only by

$$\|\alpha\| := \sum_{C \in \mathcal{M}} \max \{\alpha(C, s) \mid s \in \mathcal{E}(C)\}$$

The **normalised violation** of a Bell inequality $\langle \alpha, R \rangle$ by an empirical model $e$ is the value

$$\frac{\max\{0, B_\alpha(e) - R\}}{\|\alpha\| - R}.$$
Proposition
Let $e$ be an empirical model.
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- The normalised violation by $e$ of any Bell inequality is at most $\text{CF}(e)$. 

$e = \text{NCF}(e) + \text{CF}(e) e_{\text{NC}}$
Bell inequality violation and the contextual fraction

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- This is attained: there exists a Bell inequality whose normalised violation by $e$ is exactly $\text{CF}(e)$.

- Moreover, this Bell inequality is tight at “the” non-contextual model $e^{\text{NC}}$.

$$e = \text{NCF}(e)e^{\text{NC}} + \text{CF}(e)e^{\text{SC}}$$
Bell inequality violation and the contextual fraction

Quantifying Contextuality LP:

Find \( \mathbf{c} \in \mathbb{R}^n \)
maximising \( 1 \cdot \mathbf{c} \)
subject to \( \mathbf{M} \mathbf{c} \leq \mathbf{v}^e \)
and \( \mathbf{c} \geq \mathbf{0} \).

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Dual LP:

Find \( \mathbf{y} \in \mathbb{R}^m \) minimising \( \mathbf{y} \cdot \mathbf{v}^e \)
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\( \mathbf{Q} \)
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S. Abramsky, R. S. Barbosa, & S. Mansfield
Bell inequality violation and the contextual fraction

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computes tight Bell inequality (separating hyperplane)
Computational explorations
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Computational tools (*Mathematica* package) to:

1. calculate quantum empirical models from any (pure or mixed) state and any sets of compatible measurements
2. calculate the incidence matrix for any measurement scenario
3. quantify the degree of contextuality of any empirical model using the LP method
4. find the Bell inequality using the dual LP.
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- Equatorial measurements at angles $(\phi_1, \phi_2)$

- E.g. $(\phi_1, \phi_2) = (0, \pi/3)$ gives Bell–CHSH model

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>(0, 0)</th>
<th>(0, 1)</th>
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<td>$b_2$</td>
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The minima of the plot (maximum contextuality) occur when

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Note that these achieve Tsirelson violation of the CHSH inequality.
2. Equatorial measurements on GHZ($n$)

- $n$-partite GHZ states, given for $n > 2$ by:

$$|\psi_{\text{GHZ}(n)}\rangle = \frac{|\uparrow\rangle \otimes^n + |\downarrow\rangle \otimes^n}{\sqrt{2}}$$
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- Again, equatorial measurements on the Bloch sphere.
2. Equatorial measurements on GHZ($n$)

**Figure:** Non-contextual fraction of empirical models obtained with equatorial measurements at $\phi_1$ and $\phi_2$ on each qubit of $|\psi_{\text{GHZ}(n)}\rangle$ with: (a) $n = 3$; (b) $n = 4$. 
2. Equatorial measurements on GHZ\((n)\)

- \(n = 3\): minima of the plot reach 0 (strong contextuality) at

\[\{\phi_1, \phi_2\} \in \left\{\left\{\frac{\pi}{2}, 0\right\}, \left\{\frac{2\pi}{3}, \frac{\pi}{6}\right\}, \left\{\frac{5\pi}{6}, \frac{\pi}{3}\right\}\right\}.\]
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- General $n$: equatorial measurements at
  \[
  (\phi_1, \phi_2) \in \left\{ \left( \frac{(n + k)\pi}{2n}, \frac{k\pi}{2n} \right) \mid 0 \leq k < n \right\}
  \]
on each qubit of the $n$-partite GHZ state give rise to the strongly contextual $\text{GHZ}(n)$ model.
Further directions

- Negative Probabilities
  - Alternative relaxation of global probability distribution requirement.
  - Find quasi-probability distribution $q$ on $O_X$ such that $q|_C = e^C$.
  - . . . with minimal weight $|q| = 1 + 2\epsilon$.
  - The value $\epsilon$ provides alternative measure of contextuality.
  - Corresponds to affine decomposition $e = (1 + \epsilon)e_1 - \epsilon e_2$ with $e_1$ and $e_2$ both non-contextual.
  - Corresponding inequalities $|B_\alpha(e)| \leq R$ for cyclic measurement scenarios.

- Resource Theory
  - More than one possible measure of contextuality.
  - What properties should a good measure satisfy?
  - Operations that do not increase contextuality: relabellings, restriction, coarse-graining outcome values, tensoring, (some form of sequential composition?)
  - Towards a resource theory as for entanglement (e.g. LOCC), non-locality, . . .
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q \mid C = e^C
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Corresponds to affine decomposition

\[
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Corresponding inequalities

\[
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Cyclic measurement scenarios

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- **Signalling models**
  - Empirical data may sometimes not satisfy no-signalling (compatibility).

Given a signalling table, can we quantify amount of no-signalling and contextuality?

Similarly, we can define no-signalling fraction $\lambda_{NS} = \lambda_{NS} - (1 - \lambda_{SS})$.

Analysis of real data:

- $e_{Delft} \approx 0.0664 e_{NS} + 0.4073 e_{SC} + 0.5263 e_{NC}$
- $e_{NIST} \approx 0.0000049 e_{SS} + 0.0000281 e_{SC} + 0.9999670 e_{NC}$

First extract NS fraction, then NC fraction? Or vice-versa? Also: non-uniqueness of witnesses!

Connections with Contextuality-by-Default (Dzhafarov et al.)

Resource Theory

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    \[ e = \lambda e^{NS} - (1 - \lambda) e^{SS} \]

  ▶ Analysis of real data:

    \[ e_{Delft} \approx 0.0664 e_{SS} + 0.4073 e_{SC} + 0.5263 e_{NC} \]
    \[ e_{NIST} \approx 0.0000049 e_{SS} + 0.0000281 e_{SC} + 0.9999670 e_{NC} \]

  ▶ First extract NS fraction, then NC fraction? Or vice-versa? Also: non-uniqueness of witnesses!

  ▶ Connections with Contextuality-by-Default (Dzhafarov et al.)
Further directions

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- Signalling models
- Resource Theory
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  - Towards a resource theory as for entanglement (e.g. LOCC), non-locality, . . .
Questions...