Infinite-dimensionality in quantum foundations

W*-algebras as presheaves over matrix algebras

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Outline

Where we are, sofar

Overview of the talk

- $\mathbf{W^*\text{-}Alg}_{\mathrm{CP}}$, category of W*-algebras together with completely positive maps.
- $\mathbb{N}_{\mathrm{CP}}$, category of natural numbers n, seen as the algebra of $n \times n$ complex matrices, and completely positive maps (CP-maps) between them.
- **Set**, category of sets and functions.

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- **Theorem:** The hom-set functor

$$\mathbf{W^*\text{-}Alg}_{\mathrm{CP}}(-,=) : \mathbf{W^*\text{-}Alg}_{\mathrm{CP}}^{\mathbf{op}} \to [\mathbb{N}_{\mathrm{CP}}, \mathbf{Set}]$$

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- **What is the intuition behind this theorem?**

Algebra of complex matrices

- C*-algebra: algebra of physical observables.
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- $M_k(A)$, C*-algebra of $k \times k$ matrices whose entries are in the C*-algebra A.
- **Intuition:** as a vector space, $M_k(A)$ is $M_k \otimes A$.

Quantum channels = completely positive maps

- A linear map $f : A \to B$ is *completely positive* if it is n-positive for every $n \in \mathbb{N}$, i.e. the following map is positive for every $n \in \mathbb{N}$:

$$\begin{array}{rlcl} M_n(f): & M_n(A) & \to & M_n(B) \\ & [x_{i,j}]_{i,j \leq n} & \mapsto & [f(x_{i,j})]_{i,j \leq n} \end{array}$$

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- **Complete positivity is at the core of quantum computation**

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Examples of W*-algebras

- All finite dimensional C*-algebras.
- Every algebra of bounded operators $\mathcal{B}(H)$ on any Hilbert space H.
- Every function space $L^\infty(X)$ for any standard measure space X.
- The space $\ell^\infty(\mathbb{N})$ of bounded sequences.

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Categories of W*-algebras

- $\mathbf{W^*\text{-}Alg}_{\mathrm{CP}}$: category of W*-algebras and (normal) CP-maps
- $\mathbf{W^*\text{-}Alg}_{\mathrm{P}}$: category of W*-algebras and (normal) positive maps

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- **Argument 2**: Infinite dimensionality arise naturally in quantum field theory [Zeidler, 2008].
- **Argument 3**: The register space in a scalable quantum computer arguably has an infinite dimensional aspect [Ralph et al., 2003].
- **Argument 4**: Infinite dimensionality comes into play in Quantum PL (e.g. [Gielerak and Sawerwain, 2010, Rennela and Staton, 2015]).

Main result

- $\mathbf{W}^*\text{-}\mathbf{Alg}_{\mathrm{CP}}$, category of W*-algebras and CP-maps.
- $\mathbb{N}_{\mathrm{CP}}$, category of natural numbers n (seen as M_n) and CP-maps.
- **Set**, category of sets and functions.

- **Theorem:** The hom-set functor

$$H = \mathbf{W}^*\text{-}\mathbf{Alg}_{\mathrm{CP}}(-,=) : \mathbf{W}^*\text{-}\mathbf{Alg}_{\mathrm{CP}}^{\mathbf{op}} \to [\mathbb{N}_{\mathrm{CP}}, \mathbf{Set}]$$

is full and faithful, i.e.

$$\mathbf{W}^*\text{-}\mathbf{Alg}_{\mathrm{CP}}^{\mathbf{op}}(A,B) \cong [\mathbb{N}_{\mathrm{CP}}, \mathbf{Set}](H(A), H(B))$$

for A, B W*-algebras.

Rewording

- The hom-set functor

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- **Intuition:** In operator-theoretic categorical quantum foundations, *finite-dimensional* quantum structures can **approximate** their *infinite-dimensional* counterparts.

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Main result

Key steps of the proof

Related work

Summary

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(1) $\mathbf{W^*\text{-}Alg}_{\mathrm{CP}} \to [\mathbb{N}_{\mathrm{CP}}, \mathbf{W^*\text{-}Alg}_{\mathrm{P}}]$ is a full and faithful functor.

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(3) Natural maps in

$$[\mathbb{N}_{\mathrm{CP}}, \mathbf{Set}] \cong [\mathbf{FdC^*\text{-}Alg}_{\mathrm{CP}}, \mathbf{Set}]$$

are cone homomorphisms since $\mathbf{FdCC^*\text{-}Alg}_{\mathrm{CP}}$, category of commutative C*-algebras, is equivalent to the Lawvere theory of abstract cones.

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- **Result:** $\mathbf{W^*\text{-}Alg}_{\mathrm{CP}}(-, =) : \mathbf{W^*\text{-}Alg}_{\mathrm{CP}}^{\mathbf{op}} \to [\mathbb{N}_{\mathrm{CP}}, \mathbf{Set}]$ is full and faithful.

Variation on [Rennela and Staton, 2015]

- Full and faithful functor $M : \mathbf{W^*\text{-}Alg}_{\mathrm{CP}} \to [\mathbb{N}_{\mathrm{CP}}, \mathbf{W^*\text{-}Alg}_{\mathrm{P}}]$
  - (C*-algebra) $\mapsto$ (indexed family of C*-algebras)

$$M(A) = \{M_n(A)\}_n$$

  - (CP-map) $\mapsto$ (natural family of positive maps)

$$M(A \xrightarrow{f} B) = \{M_n(A) \xrightarrow{M_n(f)} M_n(B)\}_n$$

- (Faithfulness is obvious. Fullness is more involved and requires the use of stabilizer states.)

W*-algebras as cones

- The homset $\mathbf{W^*\text{-}Alg}_{\mathrm{P}}(A, \mathbb{C})$ of positive linear functionals of a W*-algebra A is a cone, i.e. a module for the semiring of positive reals.

- The normal positive linear functional functor

$$\mathbf{W^*\text{-}Alg}_{\mathrm{P}}(-, \mathbb{C}) : \mathbf{W^*\text{-}Alg}_{\mathrm{P}}^{\mathbf{op}} \to \mathbf{Cone}$$

  is full and faithful, where **Cone** is the category of cones and structure preserving functions between them.

(Fullness essentially comes from the fact that the closedness and completeness of the positive cone imply that every positive linear map $A_* \to \mathbb{C}$ is bounded (see e.g. [Namioka, 1957] or [Schaefer, 1966, Th. V.5.5(ii)]).)

Lawvere and the cones

- $\textbf{FdC}^*\textbf{-Alg}_{\mathrm{CP}}$, category of all finite dimensional C*-algebras and completely positive maps.

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- $\mathbf{FdCC^*\text{-}Alg}_{\mathrm{CP}}$, category of commutative finite-dimensional C*-algebras and completely positive maps.

- $\mathbf{FdCC^*\text{-}Alg}_{\mathrm{CP}}$ is equivalent to the Lawvere theory for abstract cones. [Furber and Jacobs, 2013, Prop. 4.3]

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- $\mathbf{FdCC^*\text{-}Alg}_{\mathrm{CP}}$ is equivalent to the Lawvere theory for abstract cones. [Furber and Jacobs, 2013, Prop. 4.3]
- **Consequence: Cone**, category of cones, is a full subcategory of the category $[\mathbf{FdCC^*\text{-}Alg}_{\mathrm{CP}}, \mathbf{Set}]$.

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- Link with Tobias Fritz' perspective on infinite-dimensionality? [Fritz, 2013]

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Future work

- Study precisely the construction of W*-algebras as colimits.
- Characterize which presheaves on $\mathbb{N}_{\mathrm{CP}}$ correspond to W*-algebras.
- Link with Tobias Fritz' perspective on infinite-dimensionality? [Fritz, 2013]
- Tensor product as the unique extension of the standard tensor product of matrix algebras that preserves colimits of CP-maps in each argument. [Day, 1970]

Where we are, sofar

Colimits in operator theory

- AF C*-algebras are limits of directed diagrams of finite-dimensional C*-algebras and *-homomorphisms [Bratteli, 1972].
- In a dual direction, C*-algebras and *-homomorphisms form a locally presentable category [Pelletier and Rosický, 1993].

(Quantum) PL

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  - $\mathbf{Q}$ is the category of finite sequences of finite-dimensional Hilbert spaces and trace non-increasing completely positive maps.
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- [Pagani et al., 2014] $\overline{\mathbb{N}_{\mathrm{CP}}}^{\oplus}$ (biproduct completion of $\mathbb{N}_{\mathrm{CP}}$).
  - Can we think about objects of this category as W*-algebras?

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- Boolean algebras are dense in effect algebras. [Staton and Uijlen, 2015]
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  [Flori and Fritz, 2016, Thm. 4.5]
- **Intuition:** base category as a classical perspective.
- How to study tensor products of C*/W*-algebras in this way?

Topological vector spaces

- $\mathbf{TVect}_{\mathbb{C}}$, category of topological vector spaces over $\mathbb{C}$ and continuous $\mathbb{C}$-linear maps.

- Consider a subcategory $\mathbf{V}$ of $\mathbf{TVect}_{\mathbb{C}}$ such that
  (1) $\mathbb{C} \in \mathbf{V}$ and $A \in \mathbf{V} \implies M_n(A) \in \mathbf{V}$ (closed under matrices)
  (2) $M : \mathbf{V}_{\mathrm{C}} \times \mathbb{N}_{\mathrm{Mat}} \to \mathbf{TVect}_{\mathbb{C}}$ factors through $\mathbf{V}$

- $\mathbf{V}_{\mathrm{C}}$ the closure of the category V under matrices of morphisms.

Theorem: the functor $M : \mathbf{V}_{\mathrm{C}} \to [\mathbb{N}_{\mathrm{Mat}}, \mathbf{V}]$ is full and faithful.

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$M : \mathbf{OpSpace} \to [\mathbb{N}_{\mathrm{Mat}}, \mathbf{Banach}]$ and $M : \mathbf{OpSystem} \to [\mathbb{N}_{\mathrm{Mat}}, \mathbf{OUS}]$

full and faithful functors.

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- Concrete applications? Quantum PL, contextuality, ...
  We're looking into it!