Causality and indefinite causal order in quantum theory

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Quantum theory challenges classical notions of causality

Bell scenario:

In quantum theory:

\[ p(a, b|x, y) = \text{Tr}[(E^A_{a|x} \otimes E^B_{b|y})\rho^{AB}] \]
Quantum theory challenges classical notions of causality

Bell scenario:

\[ p(a, b|x, y) = \sum_{\lambda} p(a|x, \lambda)p(b|y, \lambda)p(\lambda) \]
Quantum theory challenges classical notions of causality

Yet, **signaling** between space-like separated locations is **impossible**.

(QT respects the causal structure of space-time)

In quantum theory: 

\[ p(a|x, y) = p(a|x) , \quad p(b|x, y) = p(b|y) \]
Quantum theory challenges classical notions of causality

Yet, **signaling** between space-like separated locations is **impossible**.

(QT respects the causal structure of space-time)

Bell scenario:

A more general, genuinely quantum, notion of causality may be needed?
Quantum theory challenges classical notions of causality

The order of operations could depend on a variable in a quantum superposition:

\[(\text{indefinite causal structures?})\]

quantum SWITCH

\[
(\alpha|0\rangle + \beta|1\rangle)|\psi\rangle \rightarrow \alpha|0\rangle U^A U^B |\psi\rangle + \beta|1\rangle U^B U^A |\psi\rangle
\]

Quantum theory challenges classical notions of causality


Questions:

Can we generalize quantum theory such that a predefined causal structure is not assumed?

What new possibilities would follow from such a generalization?
Outline

• Quantum theory as an operational probabilistic theory in the circuit framework

• The axiom of causality and its meaning

• The process matrix framework for local operations without global causal structure
  - causal inequality violations
  - causal versus causally separable processes
  - dynamical causal relations

• A time-symmetric operational approach

• Quantum theory without any predefined causal structure
Operational Approach

Ludwig (1983, 1985)
A theory prescribes probabilities for the outcomes of operations.

The circuit framework for operational probabilistic theories

**Operation (test):** one use of a device with an input and an output system:

\[
\begin{array}{c}
\text{B} \\
\{M_i\} \\
\text{A}
\end{array}
\]

outcome

\[i \in O\]

Hardy, PIRSA:09060015;
Chiribella, D’Ariano, Perinotti, PRA 81, 062348 (2010) [arXiv 2009],
Chiribella, D’Ariano, Perinotti, PRA 84, 012311 (2011);
The circuit framework for operational probabilistic theories

Operation (test): one use of a device with an input and an output system:

\[ \begin{array}{c}
A \\
\{M_i\} \\
B \\
\end{array} \]

outcome 
\[ i \in O \]

a set of possible events

The circuit framework for operational probabilistic theories

*Preparations* (the input system is the trivial system $I$):

\[ \{M_i\} \]

*Measurements* (the output system is the trivial system $I$):

\[ \{L_j\} \]
The circuit framework for operational probabilistic theories

Operations can be *composed* in sequence and in parallel without forming loops:

**Sequential composition:**

$$\{N_j\}_{j \in O_2} \circ \{M_i\}_{i \in O_1} = \{L_k\}_{k \in O_1 \times O_2}$$

The circuit framework for operational probabilistic theories

Operations can be *composed* in sequence and in parallel without forming loops:

*Parallel composition:*

The circuit framework for operational probabilistic theories

**Circuit** (an acyclic composition of operations with no open wires):
The circuit framework for operational probabilistic theories

**Circuit** (an acyclic composition of operations with no open wires):

\[
\begin{align*}
\{L_k\} & \quad \{P_l\} \\
\{M_i\} & \quad \{N_j\}
\end{align*}
\]

**Probabilistic structure**

*Joint* probabilities

\[
p(i, j, k, l | \text{circuit}) \geq 0
\]

\[
\sum_{ijkl} p(i, j, k, l | \text{circuit}) = 1
\]
The circuit framework for operational probabilistic theories

Equivalently,

\[ \mathcal{A} \]

\[ \{\rho_i\} \]

\[ \{E_j\} \]

Joint probabilities

\[ p(i, j| \{\rho_i\}, \{E_j\}) \]
The circuit framework for operational probabilistic theories

An OPT is completely defined by specifying all possible operations and the probabilities for all possible circuits.

If two operations yield the same probabilities for all possible circuits they may be part of, they are deemed *equivalent*.

If two events (which may be part of different operations) yield the same probabilities for all possible circuits they may be part of, they are deemed *equivalent*.

*States*: equivalence classes of preparation events

*Effects*: equivalence classes of measurement events

*Transformations*: equivalence classes of general events from A to B
The circuit framework for operational probabilistic theories

\[ p(i, j| \{\rho_i\}, \{E_j\}) = p(\rho_i, E_j) \]

(non-contextual) function of the respective state and effect
The circuit framework for operational probabilistic theories

States are real functions on effects, and vice versa.
(elements of two dual vector spaces)
The case of standard quantum theory

System $A \rightarrow$ Hilbert space $\mathcal{H}^A$ of dimension $d_A$. 
The case of standard quantum theory

System $A \rightarrow$ Hilbert space $\mathcal{H}^A$ of dimension $d_A$.

Composite system $XY \rightarrow \mathcal{H}^X \otimes \mathcal{H}^Y$. 
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Trivial system $I \rightarrow$ 1-dimensional Hilbert space $\mathbb{C}^1$. 
The case of standard quantum theory

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Composite system $XY$ $\rightarrow$ $\mathcal{H}^X \otimes \mathcal{H}^Y$.

Trivial system $I$ $\rightarrow$ 1-dimensional Hilbert space $\mathbb{C}^1$.

Transformation from $A$ to $B$ $\rightarrow$ completely positive (CP) linear map $\mathcal{M}^{A\rightarrow B} : \mathcal{L}(\mathcal{H}^A) \rightarrow \mathcal{L}(\mathcal{H}^B)$.
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$$\mathcal{M}^{A\rightarrow B} : \mathcal{L}(\mathcal{H}^A) \rightarrow \mathcal{L}(\mathcal{H}^B)$$

(Kraus form: $\mathcal{M}^{A\rightarrow B}(\cdot) = \sum_{\alpha=1}^{d_Ad_B} K_\alpha(\cdot)K_\alpha^\dagger$, $K_\alpha : \mathcal{H}^A \rightarrow \mathcal{H}^B$)
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Operation from $A$ to $B \rightarrow$ \{\(\mathcal{M}_{i}^{A\rightarrow B}\)\}_{i \in O}

where $\Sigma_{i \in O} \mathcal{M}_{i}^{A\rightarrow B} = \overline{\mathcal{M}}^{A\rightarrow B}$ is trace preserving (CPTP).
The case of standard quantum theory

State: \( \rho^{I \rightarrow A} (\cdot) = \sum_{\alpha=1}^{d_A} |\psi_\alpha(\cdot)\rangle \langle \psi_\alpha|_A \), isomorphic to \( \rho^A = \sum_{\alpha=1}^{d_A} |\psi_\alpha\rangle \langle \psi_\alpha|_A \geq 0 \).

(non-normalized `density operator` )
The case of standard quantum theory

State: \[ \rho^{I \rightarrow A}(\cdot) = \sum_{\alpha=1}^{d_A} |\psi_{\alpha}\rangle(\cdot)\langle\psi_{\alpha}|^A, \text{ isomorphic to } \rho^A = \sum_{\alpha=1}^{d_A} |\psi_{\alpha}\rangle\langle\psi_{\alpha}|^A \geq 0. \] (non-normalized ‘density operator’)

Preparation: \( \{\rho^A_i\}_{i \in O} \), where \( \sum_{i \in O} \text{Tr}(\rho^A_i) = 1 \).
The case of standard quantum theory

State: $\rho^{I\rightarrow A}(\cdot) = \sum_{\alpha=1}^{d_A} |\psi_\alpha(\cdot)\rangle\langle\psi_\alpha|_A^A$, isomorphic to $\rho^A = \sum_{\alpha=1}^{d_A} |\psi_\alpha\rangle\langle\psi_\alpha|_A^A \geq 0$.

(Non-normalized ‘density operator’)

Preparation: $\{\rho^A_i\}_{i \in O}$, where $\sum_{i \in O} \text{Tr}(\rho^A_i) = 1$.

Effect: $E^{A\rightarrow I}(\cdot) = \sum_{\alpha=1}^{d_A} \langle \phi_\alpha(\cdot) | \phi_\alpha \rangle^A_\cdot \quad E^A = \sum_{\alpha=1}^{d_A} |\phi_\alpha\rangle\langle\phi_\alpha|_A^A \geq 0$
The case of standard quantum theory

State: \[ \rho_{I\to A}(\cdot) = \sum_{\alpha=1}^{d_A} |\psi_\alpha\rangle\langle\psi_\alpha|^A, \] isomorphic to \[ \rho^A = \sum_{\alpha=1}^{d_A} |\psi_\alpha\rangle\langle\psi_\alpha|^A \geq 0. \]

(non-normalized ‘density operator’)

Preparation: \[ \{\rho_i^A\}_{i\in O}, \] where \[ \sum_{i\in O} \text{Tr}(\rho_i^A) = 1. \]

Effect: \[ E^A_{I\to I}(\cdot) = \sum_{\alpha=1}^{d_A} \langle \phi_\alpha | (\cdot) | \phi_\alpha \rangle^A \]

Measurement: \[ \{E_j^A\}_{j\in Q}, \] where \[ \sum_{j\in Q} E_j^A = 1^A. \]

[Positive operator-valued measure (POVM)]
The case of standard quantum theory

State: \( \rho^{I\rightarrow A}(\cdot) = \sum_{\alpha=1}^{d_A} |\psi_\alpha\rangle \langle \psi_\alpha|^A \), isomorphic to \( \rho^A = \sum_{\alpha=1}^{d_A} |\psi_\alpha\rangle \langle \psi_\alpha|^A \geq 0 \).

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Effect: \( E^{A\rightarrow I}(\cdot) = \sum_{\alpha=1}^{d_A} \langle \phi_\alpha | (\cdot) | \phi_\alpha \rangle^A \quad \leftrightarrow \quad E^A = \sum_{\alpha=1}^{d_A} |\phi_\alpha\rangle \langle \phi_\alpha|^A \geq 0 \)

Measurement: \( \{E_j^A\}_{j \in Q} \), where \( \sum_{j \in Q} E_j^A = 1^A \).

[Positive operator-valued measure (POVM)]

Main probability rule: \( p(\rho^{I\rightarrow A}, E^{A\rightarrow I}) = E^{A\rightarrow I} \circ \rho^{I\rightarrow A} = \text{Tr}[\rho^A E^A] \)
The case of standard quantum theory

State: \( \rho^{I \to A}(\cdot) = \sum_{\alpha=1}^{d_A} |\psi_\alpha(\cdot)\rangle\langle \psi_\alpha|^A \), isomorphic to \( \rho^A = \sum_{\alpha=1}^{d_A} |\psi_\alpha\rangle\langle \psi_\alpha|^A \geq 0 \).

(non-normalized ‘density operator’)

Preparation: \( \{\rho_i^A\}_{i \in O} \), where \( \sum_{i \in O} \text{Tr}(\rho_i^A) = 1 \).

(not a natural isomorphism!)

Effect: \( E^{A \to I}(\cdot) = \sum_{\alpha=1}^{d_A} \langle \phi_\alpha(\cdot) | \phi_\alpha \rangle^A \)

\( E^A = \sum_{\alpha=1}^{d_A} |\phi_\alpha\rangle\langle \phi_\alpha|^A \geq 0 \)

Measurement: \( \{E_j^A\}_{j \in Q} \), where \( \sum_{j \in Q} E_j^A = 1_A^A \).

[Positive operator-valued measure (POVM)]

Main probability rule: \( p(\rho^{I \to A}, E^{A \to I}) = E^{A \to I} \circ \rho^{I \to A} = \text{Tr}[\rho^A E^A] \)
The marginal probabilities of the preparation, \( p(\{\rho_i\}, \{E_j\}) \), are independent of the measurement:

\[
p(i, j \mid \{\rho_i\}, \{E_j\}) = p(\rho_i, E_j) = Tr(\rho_i^A E_j^A) \quad \text{in QT}
\]

The causality axiom

\[
\forall \{\rho_i\}, \{E_j\}, \{F_k\}
\]

\[
p(i \mid \{\rho_i\}, \{E_j\}) = p(i \mid \{\rho_i\}, \{F_k\})
\]

‘No signalling from the future’
The causality axiom

Chiribella, D'Ariano, Perinotti, PRA 81, 062348 (2010), PRA 84, 012311 (2011):

Some properties of causal theories:

• There is a unique deterministic effect (in quantum theory, $\mathbb{1}^A$).

• Conditioned operations are possible

• If a causal theory is not deterministic and the set of states is closed, the set of states is **convex**.
What is the axiom of causality about?
What is an operation?
What is an operation?


Two ideas:
What is an operation?
Idea 1. The ‘closed-box’ assumption

All correlations between the events in the boxes are due to information exchange through the wires.

(The concept of circuit formalizes the idea of information exchanged via systems)
What is an operation?

Idea 1. The ‘closed-box’ assumption

All correlations between the events in the boxes are due to information exchange through the wires.

(The concept of circuit formalizes the idea of information exchanged via systems)

An operation could be realized inside an isolated box.
Imagine Alice who chooses to perform one out of many possible operations $\{M_{j\alpha}\}$ with probability $p(\alpha)$ inside a closed box.

- If Charlie doesn’t known the choice of Alice, he can say that the operation is $\{\{p(\alpha_1)M_{j\alpha_1}\},\{p(\alpha_2)M_{j\alpha_2}\},\cdots\}$.

- If he learns that Alice has chosen $\alpha$, he can say that the operation is $\{M_{j\alpha}\}$.
  (This is consistent with the Bayesian update of the probabilities of a circuit.)

→ A subset of the possible events in an operation defines another operation.
Imagine Alice who chooses to perform one out of many possible operations \( \{ M^\alpha_{j\alpha} \} \) with probability \( p(\alpha) \) inside a closed box.

- If Charlie doesn’t know the choice of Alice, he can say that the operation is \( \{ \{ p(\alpha_1)M^\alpha_{j\alpha_1} \} , \{ p(\alpha_2)M^\alpha_{j\alpha_2} \} , \cdots \} \).

- If he learns that Alice has chosen \( \alpha \), he can say that the operation is \( \{ M^\alpha_{j\alpha} \} \).

But not all subsets of events are considered valid operations!

Why?
What is an operation?

**Intuition:** a valid operation can be *chosen* by the experimenter, but an arbitrary subset of its outcomes cannot.
What is an operation?

Intuition: a valid operation can be chosen by the experimenter, but an arbitrary subset of its outcomes cannot.

How do we formalize this?
What is an operation?

**Intuition:** a valid operation can be *chosen* by the experimenter, but an arbitrary subset of its outcomes cannot.

**How do we formalize this?**

**A guess:** define that the ‘choice’ of operation is independent of past events.
What is an operation?

**Intuition:** a valid operation can be *chosen* by the experimenter, but an arbitrary subset of its outcomes cannot.

**How do we formalize this?**

**A guess:** define that the ‘choice’ of operation is independent of past events.

**Problem:** this would mean that the causality axiom is a *definition* and not an axiom. However, the axiom seems to express a non-trivial physical constraint.
What is an operation?

**Intuition:** a valid operation can be *chosen* by the experimenter, but an arbitrary subset of its outcomes cannot.

**How do we formalize this?**

Idea 2: The ‘no post-selection’ criterion:

The ‘choice’ of operation can be known *before* the time of the input system, irrespectively of future events.

Under this criterion, the causality axiom expresses a *nontrivial constraint*. 
What is an operation?

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How do we formalize this?

**Idea 2:** The ‘no post-selection’ criterion:

The ‘choice’ of operation can be known *before* the time of the input system, irrespectively of future events.

Under this criterion, the causality axiom expresses a *nontrivial constraint*.

The very concept of operation is *time-asymmetric*!
Does the property of causality imply an actual physical asymmetry?

(Note: The formal asymmetry does not automatically imply a physical asymmetry because the very concept of operation is asymmetric.)

It actually does - The ‘pre-selected’ operations in the reverse time direction are all post-selected operations in the forward direction.

These time-reversed operations do not obey the causality axiom.

Physics under time reversal is not described by the usual quantum theory.
To summarize, in the circuit framework, a notion of time is presumed.

Events are equipped with a partial (causal) order coming from the circuit composition – one operation precedes another if there is a directed path from the former to the latter through the circuit.
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Events are equipped with a partial (causal) order coming from the circuit composition – one operation precedes another if there is a directed path from the former to the latter through the circuit.

Can we understand time and causal structure from more primitive concepts?

(e.g., signaling from Alice to Bob $\rightarrow$ Alice is in the past of Bob)
A local experiment can exchange information with the outside world only via the input and output systems.

1) A system enters the lab.
2) A setting $S^A$ is chosen.
3) An outcome $O^A$ is obtained.
4) A system exits the lab.

The process framework

No assumption of global causal order between the local experiments.

The process framework

Alice

Bob

‘Process’
(catalogue of probabilities)

\[ p(o^A, o^B, \ldots | s^A, s^B, \ldots) \]
Quantum processes

Local descriptions agree with quantum mechanics

Transformations = completely positive (CP) maps

Kraus representation:

$$\mathcal{M}_j : \mathcal{L}(\mathcal{H}^1) \rightarrow \mathcal{L}(\mathcal{H}^2)$$

Completenss relation:

$$\sum_j \sum_k E_{jk}^\dagger E_{jk} = I$$
Assumption 1: The probabilities are functions of the local CP maps,

\[ P(\mathcal{M}^A_{j^A}, \mathcal{M}^B_{j^B}, \cdots) \]

Local validity of QM \( \rightarrow \) \( P(\mathcal{M}^A, \mathcal{M}^B, \cdots) \) is linear in \( \mathcal{M}^A, \mathcal{M}^B, \cdots \)
Choi-Jamiołkowski isomorphism

$\mathcal{M} : \mathcal{L}(\mathcal{H}^1) \rightarrow \mathcal{L}(\mathcal{H}^2)$

$M^{12} \in \mathcal{L}(\mathcal{H}^1) \otimes \mathcal{L}(\mathcal{H}^2)$
Choi-Jamiołkowski isomorphism

\[ M : \mathcal{L}(\mathcal{H}^1) \to \mathcal{L}(\mathcal{H}^2) \quad \leftrightarrow \quad M^{12} \in \mathcal{L}(\mathcal{H}^1) \otimes \mathcal{L}(\mathcal{H}^2) \]

\[ M^{12} := [I \otimes M(|\Phi^+\rangle\langle\Phi^+|)]^T \]

\[ |\Phi^+\rangle = \sum_i |i\rangle|i\rangle \in \mathcal{H}^1 \otimes \mathcal{H}^1 \]
Choi-Jamiołkowski isomorphism

\[ M : L(\mathcal{H}^1) \rightarrow L(\mathcal{H}^2) \quad \leftrightarrow \quad M^{12} \in L(\mathcal{H}^1) \otimes L(\mathcal{H}^2) \]

\[ M(\rho^1) = [\text{Tr}_1(\rho^1 M^{12})]^T \]
The process matrix

\[ P(M^A_{jA}, M^B_{jB}, \cdots) = \text{Tr} \left[ W^{A_1A_2B_1B_2\cdots} \left( M^{A_1A_2}_{jA} \otimes M^{B_1B_2}_{jB} \otimes \cdots \right) \right] \]
The process matrix

\[ P(\mathcal{M}_j^A, \mathcal{M}_j^B, \cdots) = \text{Tr} \left[ W^{A_1A_2B_1B_2\cdots} \left( \mathcal{M}_j^{A_1A_2} \otimes \mathcal{M}_j^{B_1B_2} \otimes \cdots \right) \right] \]

Similar to Born‘s rule but can describe signalling!
The process matrix

Assumption 2: The parties can share entangled input ancillas.

Conditions on $W$:

1. Non-negative probabilities: $W^{A_1A_2B_1B_2\ldots} \geq 0$

2. Probabilities sum up to 1:
   $$\text{Tr} \left[ W^{A_1A_2B_1B_2\ldots} \left( M^{A_1A_2} \otimes M^{B_1B_2} \otimes \cdots \right) \right] = 1$$
   on all CPTP maps $M^{A_1A_2}$, $M^{B_1B_2}$, ...

Note: $M^{A_1A_2}$ is CPTP iff $M^{A_1A_2} \geq 0$, $\text{Tr}_{A_2} M^{A_1A_2} = 1^{A_1}$.
The process matrix

An alternative formulation as a second-order operation:

[Quantum supermaps, Chiribella, D’Ariano, and Perinotti, EPL 83, 30004 (2008)]
Terms appearing in a process matrix

\[ W^{A_1A_2B_1B_2C_1C_2 \cdots} = \sum_{i,j,k,l,m,n} W_{ijklmn} \sigma^A_i \otimes \sigma^B_j \otimes \sigma^K_i \otimes \sigma^B_i \otimes \sigma^C_j \otimes \sigma^C_j \otimes \cdots \]

Hilbert-Schmidt basis: Hermitian \( \{ \sigma^X_\mu \}_{\mu=0}^{d_X^2-1} \), where \( \sigma^X_0 = 1^X \), \( \text{Tr} \sigma^X_\mu \sigma^X_\nu = d_X \delta_{\mu\nu} \)

**Proposition:** \( W^{A_1A_2B_1B_2\cdots} \) is a valid process matrix iff

1) \( W^{A_1A_2B_1B_2\cdots} \geq 0 \)

2) In addition to the identity, it contains only terms with a non-trivial \( \sigma \) on \( X_1 \) and \( 1 \) on \( X_2 \) for some party \( X \in \{ A, B, C, \cdots \} \).
Example: bipartite case

\[ W^{A_1A_2B_1B_2} = \sum_{\mu_1, \ldots, \mu_4} a_{\mu_1 \ldots \mu_4} \sigma_{\mu_1}^{A_1} \otimes \ldots \otimes \sigma_{\mu_4}^{B_2} \]

\[ \sigma_i^{A_1} \otimes \mathbb{1}_{\text{rest}} \quad \text{type } A_1 \]

\[ \sigma_i^{A_1} \otimes \sigma_j^{A_2} \otimes \mathbb{1}_{\text{rest}} \quad \text{type } A_1A_2 \]

\[ \ldots \]

A valid process matrix:

\[ W^{A_1A_2B_1B_2} \geq 0 \]

and contain only the identity term plus terms of type

\[ A_1, B_1, A_1B_1, A_2B_1, A_1B_2, A_1A_2B_1, A_1B_1B_2 \]
Example: bipartite state

\[ W_{A_1 A_2 B_1 B_2} = \rho^{A_1 B_1} \otimes 1^{A_2 B_2} \]
Example: channel $B \rightarrow A$

\[
W^{A_1 A_2 B_1 B_2} = \mathbb{1}^{A_2} \otimes (C^{A_1 B_2})^T \otimes \rho^{B_1}
\]
Example: channel with memory $A \rightarrow B$

(The most general possibility compatible with no signalling from B to A!)

$$W^{A_1 A_2 B_1 B_2} = W^{A_1 A_2 B_1} \otimes \mathbb{I}^{B_2}$$
Causal order

causal future

causal past

causal elsewhere
(Strict) partial order $\prec$:

1) *irreflexivity*

   not $X \prec X$.

2) *transitivity*

   if $X \prec Y$ and $Y \prec Z$, then $X \prec Z$.

3) *antisymmetry*

   if $X \prec Y$, then not $Y \prec X$. 

---

Causal order

- **causal future**
  - B
  - (B $\succ$ A)
  - (D $\not\prec$ A)

- **causal past**
  - C
  - (C $\prec$ A)

- **causal elsewhere**
  - A

Nodes A, B, C, and D are labeled accordingly.
Causal order

causal future

causal elsewhere

D

causal past

C

Signaling is possible

B

A
Causal order

Signaling is impossible
Causal order

causal future

causal elsewhere

Signaling is impossible
Causal order

Notation: \[ A \ngeq B \]

Alice is *not* in the causal past of Bob (hence, Alice cannot signal to Bob)

In a causal scenario, at least one of \((A \ngeq B)\) or \((B \ngeq A)\) must be true.

→ Alice cannot signal to Bob or Bob cannot signal to Alice.
Bipartite processes with causal realization

\[ W^{A \not\rightarrow B} \] – no signalling from A to B \( (\text{ch. with memory from B to A}) \)

\[ W^{B \not\rightarrow A} \] – no signalling from B to A \( (\text{ch. with memory from A to B}) \)
Bipartite processes with causal realization

\[ W^{A \not\to B} \] – no signalling from A to B (ch. with memory from B to A)

\[ W^{B \not\to A} \] – no signalling from B to A (ch. with memory from A to B)

More generally, we may conceive causally separable processes (probabilistic mixtures of fixed-order processes):

\[
W_{\text{cs}}^{A_1A_2B_1B_2} = qW^{A \not\to B} + (1 - q)W^{B \not\to A}
\]
Bipartite processes with causal realization

$W_{A \not\rightarrow B}$ – no signalling from A to B (ch. with memory from B to A)

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More generally, we may conceive causally separable processes (probabilistic mixtures of fixed-order processes):

$$W_{cs}^{A_1A_2B_1B_2} = qW^{A \not\rightarrow B} + (1 - q)W^{B \not\rightarrow A}$$

Are all process matrices causally separable?
Their goal is to maximize:

\[ p_{\text{succ}} = \frac{1}{2}[P(x = b | b' = 0) + P(y = a | b' = 1)] \]
Causally ordered situation

Case \( B \neq A \)
Causally ordered situation

Case \( B \not\preceq A \)

Global Time

\[ P(x = b|b' = 0) = 1/2 \]
Causally ordered situation

Case $B \neq A$

$P(y = a | b' = 1) = 1$

$P(x = b | b' = 0) = 1/2$
Causally ordered situation

Case \( B \not\equiv A \)

Global Time

\[
P(y = a | b' = 1) = 1
\]

\[
P(x = b | b' = 0) = 1/2
\]

\[
p_{\text{succ}} = \frac{1}{2} [P(x = b | b' = 0) + P(y = a | b' = 1)] \leq \frac{3}{4}
\]
A causal inequality

Definite causal order →

\[ p_{succ} = \frac{1}{2} [P(x = b|b' = 0) + P(y = a|b' = 1)] \leq \frac{3}{4} \]
A non-causal process

Can violate the inequality with $p_{suc} = \frac{2 + \sqrt{2}}{4} > \frac{3}{4}$.

\[ W^{A_1 A_2 B_1 B_2} = \frac{1}{4} \left[ 1 + \frac{1}{\sqrt{2}} \left( \sigma_z^{A_2} \sigma_z^{B_1} + \sigma_z^{A_1} \sigma_x^{B_1} \sigma_z^{B_2} \right) \right] \]

two-level systems

The operations of Alice and Bob do not occur in a definite order!

A causally non-separable situation

Alice always measures in the z basis and encodes the bit in the z basis

Alice’s CP map: $$|z_x\rangle\langle z_x|^A_1 \otimes |z_a\rangle\langle z_a|^A_2 \quad x, a = \pm 1$$

If Bob wants to receive (b'=1), he measures in the z basis

Channel from Alice to Bob

Not seen by Bob

Bob receives the state

$$\widetilde{W}^{B_1B_2} = \frac{1}{2} \left( \mathbb{I} + a \frac{1}{\sqrt{2}} \sigma_z^{B_1} \right)$$

He can read Alice’s bit with probability

$$P(y = a|b' = 1) = \frac{2+\sqrt{2}}{4}$$
A causally non-separable situation

If Bob wants to send \( \mathbf{b}' = 0 \), he measures in the x basis and encodes in the z basis conditioned on his outcome

Bob's CP map: \[ |x_y\rangle \langle x_y|^{B_1} \otimes |z_{by}\rangle \langle z_{by}|^{B_2} \quad y, b = \pm 1 \]

\[
W^{A_1A_2B_1B_2} = \frac{1}{4} \left[ 1 + \frac{1}{\sqrt{2}} \left( \sigma^{A_2}_z \otimes \sigma^{B_1}_z + \sigma^{A_1}_z \otimes \sigma^{B_1}_x \otimes \sigma^{B_2}_z \right) \right]
\]

\[ \langle x_\pm | \sigma^z | x_\pm \rangle = 0 \]

Not seen by Bob

Channel from Bob to Alice, correlated with Bob's outcome

Alice receives the state

\[
\widetilde{W}^{A_1A_2} = \frac{1}{2} \left( 1 + b \frac{1}{\sqrt{2}} \sigma^{A_1}_z \right)
\]

She can read Bob's bit with probability

\[ P(x = b|b' = 0) = \frac{2 + \sqrt{2}}{4} \]
Other causal inequalities and violations

**Simplest bipartite inequalities:**

Branciard, Araujo, Feix, Costa, Brukner, NJP 18, 013008 (2016)

**Multiparite inequalities:**

- violation with perfect signaling


- violation by classical local operations:

Baumeler, Feix, and Wolf, PRA 90, 042106 (2014)

Baumeler and Wolf, NJP 18, 013036 (2016)

**Biased version of the original inequality:**

Can non-causal processes be realized physically?
Can non-causal processes be realized physically?

Not *a priori* impossible!

From the outside the experiment may still agree with standard unitary evolution in time.
The quantum switch

Chiribella, D’Ariano, Perinotti and Valiron, arXiv:0912.0195, PRA 2013

The *tripartite* process is not causally separable!

\[ W^{A_1A_2B_1B_2C_1C_2} = |W\rangle\langle W|^{A_1A_2B_1B_2C_1C_2} \]

M. Araujo et al., NJP 17, 102001 (2015)
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Yet, it cannot violate causal inequalities…

M. Araujo et al., NJP 17, 102001 (2015)
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Yet, it cannot violate causal inequalities…

M. Araujo et al., NJP 17, 102001 (2015)
Advantage in black-box discrimination

Charlie can find with certainty whether two gates commute or anti-commute, even though each gate is used only once.

Advantage in black-box discrimination

Causal witness:

$$\text{tr}[S \ W^{\text{sep}}] \geq 0$$


Advantage in black-box discrimination

Computations with multipartite-SWITCH


Communication complexity:

Formal theory of causality for processes

O. O. and C. Giarmatzi, arXiv:1506.05449

What constraints on the correlations does causality imply?
The causal order can be both *random* and *dynamical*

E.g., the operation at A could influence the order in which B and C happen in A’s future.
The causal order can be both *random* and *dynamical*.

E.g., the operation at A could influence the order in which B and C happen in A’s future.
Device-independent definition of causality

O. O. and C. Giarmatzi, arXiv:1506.05449

A notion of causality should:

• have a universal expression (which implies the multipartite case)

• allow of dynamical causal order (a given event can influence the order of other events in its future)

• capture our intuition of causality
Device-independent definition of causality

O. O. and C. Giarmatzi, arXiv:1506.05449

General process: \( W^{A,B,...} \equiv \{ P(o^A, o^B, ... | s^A, s^B, ... ) \} \)

**Intuition:** The probability for a set of events to occur outside of the causal future of Alice and for these events to have a particular causal configuration with Alice is independent of the choice of setting of Alice.
A process is **causal** iff there exists a probability distribution

\[
P(\kappa(A, B, \cdots), o^A, o^B, \cdots | s^A, s^B, \cdots)
\]

where \( \kappa(A, B, \cdots) \) is a partial order, such that for every party, e.g., \( A \), and every subset \( X, Y, \cdots \) of the other parties,

\[
P(\kappa(A, X, Y, \cdots), A \not\preceq X, A \not\preceq Y, \cdots, o^X, o^Y, \cdots | s^A, s^B, \cdots) = P(\kappa(A, X, Y, \cdots), A \not\preceq X, A \not\preceq Y, \cdots, o^X, o^Y, \cdots | s^B, \cdots).
\]
Structure of causal processes

Consider \( W^{1, \cdots, n} \equiv W^{A, B} \)

\( A = \{1, \cdots, k\} \)
\( B = \{k + 1, \cdots, n\} \)

If no signaling from \( B \) to \( A \) exists, there exists a reduced process \( W^A \)

\[
p(o^1, \cdots, o^k | s^1, \cdots, s^n) = p(o^1, \cdots, o^k | s^1, \cdots, s^k)
\]
Structure of causal processes

O. O. and C. Giarmatzi, arXiv:1506.05449

Consider

\[ W^{1,\cdots,n} \equiv W^{A,B} \]

\[ A = \{1, \cdots, k\} \]

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If no signaling from \( B \) to \( A \) exists, there exists a reduced process \( W^A \)

\[ W^{A,B} \equiv W^{B|A} \circ W^A \]

conditional process
Structure of causal processes

O. O. and C. Giarmatzi, arXiv:1506.05449

Consider

\[ W_{1, \ldots, n} \equiv W^{A, B} \]

\[ A = \{1, \cdots, k\} \]
\[ B = \{k + 1, \cdots, n\} \]

If no signaling from \( B \) to \( A \) exists, there exists a reduced process \( W^A \)

\[ W^{A, B} \equiv W^{B|A} \circ W^A \]

\[
p(o_1, \cdots, o_n|s_1, \cdots, s_n) = p(o_{k+1}, \cdots, o_n|s_{k+1}, \cdots, s_n; s_1, o_1, \cdots, s^k, o_k) p(o_1, \cdots, o_k|s_1, \cdots, s^k)\]
Theorem (canonical causal decomposition):

\[ W_{1,\cdots,n}^c = \sum_{i=1}^{n} q_i W^{(1,\cdots,i-1,i+1,\cdots,n) \neq i}, \quad q_i \geq 0 \]

where

\[ W^{(1,\cdots,i-1,i+1,\cdots,n) \neq i} = W_{c,1,\cdots,i-1,i+1,\cdots,n|i} \circ W^i \]

(iterative formulation)

Describes causal ‘unraveling’ of the events in the process.
Structure of causal processes

O. O. and C. Giarmatzi, arXiv:1506.05449

**Theorem** (canonical causal decomposition):

\[ W_{c}^{1,\ldots,n} = \sum_{i=1}^{n} q_{i} W^{(1,\ldots,i-1,i+1,\ldots,n) \neq i}, \quad q_{i} \geq 0 \]

where

\[ W^{(1,\ldots,i-1,i+1,\ldots,n) \neq i} = W_{c}^{1,\ldots,i-1,i+1,\ldots,n|i} \circ W^{i} \]

( iterative formulation )

Causal correlations form polytopes! [For the bipartite case, see Branciard et al., NJP 18, 013008 (2016)]
Example of a causal inequality which is a facet:

Guess Your Neighbour’s Input (GYNI) game

\[ p(x, y) = p(x)p(y), \quad p(x) = 1/2, \quad p(y) = 1/2 \]

**Goal:** maximize \( p(a = y, b = x) \)

Example of a causal inequality which is a facet:

Guess Your Neighbour’s Input (GYNI) game

Causal order \( \kappa(A, B) = [A \neq \neq B] \)

\[
p(x, y) = p(x)p(y), \quad p(x) = 1/2, \quad p(y) = 1/2
\]

\[
p(a = y, b = x) \leq 1/2
\]

Example of a causal inequality which is a facet:

Guess Your Neighbour’s Input (GYNI) game

Causal order \( \kappa(A, B) = [A < B] \)

\[
p(x, y) = p(x)p(y), \quad p(x) = \frac{1}{2}, \quad p(y) = \frac{1}{2}
\]

\[
p(a = y, b = x) \leq \frac{1}{2}
\]
Example of a causal inequality which is a facet:
Guess Your Neighbour’s Input (GYNI) game

Causal order \( \kappa(A, B) = [B < A] \)

\[ p(x, y) = p(x)p(y), \quad p(x) = 1/2, \quad p(y) = 1/2 \]

\[ p(a = y, b = x) \leq 1/2 \]

Example of a causal inequality which is a facet:

Guess Your Neighbour’s Input (GYNI) game

\[
p(x, y) = p(x)p(y), \quad p(x) = 1/2, \quad p(y) = 1/2
\]

There exists a process matrix which allow \(p(a = y, b = x) > 1/2\).

A quantum process is called **causally separable** iff it can be written in a canonical causal form

\[ W_{c_i}^{1, \ldots, n} = \sum_{i=1}^{n} q_i W^{(1, \ldots, i-1, i+1, \ldots, n) \neq i}, \quad q_i \geq 0 \]

where

\[ W^{(1, \ldots, i-1, i+1, \ldots, n) \neq i} = W_{c}^{1, \ldots, i-1, i+1, \ldots, n|i} \circ W^i \]

with every process in this decomposition being a valid quantum process.

(analogy with Bell local and separable quantum states)

Agrees with the bipartite concept

\[ W^{A_1 A_2 B_1 B_2} = q W^{B \neq A} + (1 - q) W^{A \neq B} \]
Causal but causally nonseparable process

Chiribella, D’Ariano, Perinotti and Valiron, arXiv:0912.0195, PRA 2013

The *tripartite* process is not causally separable!

\[
W^{A_1A_2B_1B_2C_1C_2} = |W\rangle\langle W|^{A_1A_2B_1B_2C_1C_2}
\]

Yet, it cannot violate causal inequalities…

M. Araujo et al., NJP 17, 102001 (2015)
Causality and causal separability are different in the bipartitit case too


Example where \( W^{A_1A_2B_1B_2} \) is not causally separable,

but \( (W^{A_1A_2B_1B_2})^{T_{B_1B_2}} \) is causally separable.

(The two have the same statistics on local quantum operations.)
Non-causality can be *activated* by entanglement

O. O. and C. Giarmatzi, arXiv:1506.05449

Example where \( \mathcal{W}^{A_1A_2B_1B_2C_2} \) is causally separable (and hence causal),

but \( \mathcal{W}^{A_1A_2B_1B_2C_2} \otimes |\phi^+\rangle \langle \phi^+|^{B'_1C'_1} \) is non-causal.

\[
\mathcal{W}^{A_1A_2B_1B_2C_2} = \frac{1}{4} \left( \mathbb{1}^{A_1A_2B_1B_2C_2} + \frac{1}{\sqrt{2}} \sigma_z^{A_1} \sigma_z^{B_1} \sigma_z^{B_2} \sigma_x^{C_2} + \frac{1}{\sqrt{2}} \sigma_z^{A_2} \sigma_z^{B_1} \sigma_z^{C_2} \right)
\]
Non-causality can be *activated* by entanglement

O. O. and C. Giarmatzi, arXiv:1506.05449

Example where $W^{A_1A_2B_1B_2C_2}$ is causally separable (and hence causal),

$$W^{A_1A_2B_1B_2C_2} \otimes |\phi^+\rangle\langle \phi^+|^{B_1'C_1'}$$

is non-causal.

One may expect that physically relevant processes are *extensible!*

More natural concepts of interest:

Extensibility causal (EC)  
Extensibility causally separable (ECS)  

(the property does not change under extension with ancilla)
Some properties of EC and ECS processes

1) In the bipartite case, ECS = causally separable.

→ ECS is another possible multipartite generalization of the bipartite concept

\[ W^{A_1A_2B_1B_2} = qW^{B \neq A} + (1 - q)W^{A \neq B}. \]

2) EC ≠ ECS (tripartite example: the quantum switch).

The bipartite case is an open problem.

3) In the bipartite case, C ≠ EC either (Feix et al).
A protocol:

1. Prepare a quantum register in some quantum state.

2. Perform a quantum operation on the register.

3. Depending on the outcome, choose which party is first and which subsystem of the register will be his/her input system.

4. After the first party operates, perform a quantum operation on the transformed register.

5. Depending on the outcome, choose which party is second and which subsystem of the register is his/her input system.

6. Continue analogously until all parties are used.
A protocol:

1. Prepare a quantum register in some quantum state.
2. Perform a quantum operation on the register.
3. Depending on the outcome, choose which party is first and which subsystem of the register will be his/her input system.
4. After the first party operates, perform a quantum operation on the transformed register.
5. Depending on the outcome, choose which party is second and which subsystem of the register is his/her input system.
6. Continue analogously until all parties are used.

Similar to classically controlled quantum Turing machine [Knill (1996), Valiron-Selinger (2005)].
The processes realizable within this paradigm are ECS.

Conjecture: The reverse also holds: CCQC = ECS.

(certainly holds in the bipartite case)
What we know at present

(a) Multipartite case.

(b) Bipartite case.
Outlook

• Two conjectures:
  1) ECS = classically controlled quantum circuits (CCQC)?
  2) EC = quantum controlled quantum circuits (QCQC)?

• What is the structure of CCQC and QCQC process matrices?

• Are there physically admissible processes that are non-causal?

• What are the information processing powers of these classes?

• Causal inference for dynamical and quantum causal relations?
Related work

• Classical causal inference (Pearl, CUP 2009) in the context of quantum theory:


• Another notion of ‘indefinite causal structures’:

  Ried, Spekkens, ... (in preparation)

• Quantum and GPT generalizations of classical causal inference:

  ...
The process framework still assumes time locally, and it is time-asymmetric.

Could we relax the assumption of time also locally?
Idea 2. No post-selection

The ‘choice’ of operation can be known before the operation is applied

(Underlies the interpretation that an operation can be ‘chosen’.)
Proposal: drop the ‘no post-selection’ criterion

Operation =

description of the possible events in a box conditional on local information

Joint probabilities:

\[ p(i, j) = \frac{Tr(\rho_i E_j)}{Tr(\bar{\rho} \bar{E})} \]

where

\[ \bar{\rho} = \sum_i \rho_i, \quad Tr(\bar{\rho}) = 1 \]
\[ \bar{E} = \sum_j E_j, \quad Tr(\bar{E}) = d \]

The basic probability rule.

[O.O. and N. Cerf, Nature Phys. 11, 853 (2015)]

[Joint probabilities:]

\[ \{E_j\} \]
\[ \{\rho_i\} \]

[The basic probability rule.]

[Time-symmetric quantum theory]

New states and effects

*States* (equivalent preparation events): \((\rho, \bar{\rho})\), where \(0 \leq \rho \leq \bar{\rho},\ Tr(\bar{\rho}) = 1\).

*Effects* (equivalent measurement events): \((E, \bar{E})\), where \(0 \leq E \leq \bar{E},\ Tr(\bar{E}) = d\).

Joint probabilities:

\[
p[(\rho, \bar{\rho}), (E, \bar{E})] = \frac{Tr(\rho E)}{Tr(\bar{\rho} \bar{E})}, \quad Tr(\bar{\rho} \bar{E}) \neq 0
\]

\[
= 0, \quad Tr(\bar{\rho} \bar{E}) = 0
\]

States can be thought of as functions on effects and vice versa.
**New states and effects**

**States** (equivalent preparation events): \((\rho, \overline{\rho})\), where \(0 \leq \rho \leq \overline{\rho}, \text{Tr}(\overline{\rho}) = 1\).

**Effects** (equivalent measurement events): \((E, \overline{E})\), where \(0 \leq E \leq \overline{E}, \text{Tr}(\overline{E}) = d\).

Joint probabilities: \(p[(\rho, \overline{\rho}), (E, \overline{E})] = \frac{\text{Tr}(\rho E)}{\text{Tr}(\overline{\rho} \overline{E})}, \quad \text{Tr}(\overline{\rho} \overline{E}) \neq 0 \)

\[= 0, \quad \text{Tr}(\overline{\rho} \overline{E}) = 0\]

The set of states (effects) is not closed under convex combinations!

States can be thought of as functions on effects and vice versa.
General operations

General operations: collections of CP maps \( \{M_j\} \), s.t. \( \text{Tr}(\sum_j M_j(\frac{I}{d_A})) = 1 \).

Transformations: \((M, \bar{M})\), where \( 0 \leq M \leq \bar{M} \), \( \text{Tr}(\bar{M}(\frac{I}{d_A})) = 1 \).
Time reversal symmetry

Example:

\[ p(i, j, k) = \frac{Tr(E_k^B M_j^{A\rightarrow B} (\rho_i^A))}{Tr(E^B \bar{M}^{A\rightarrow B} (\bar{\rho}^A))} \]
Time reversal symmetry

Example:

\[ p(i, j, k) = \frac{\text{Tr}(F_i^A N_j^{B \rightarrow A} (\sigma_k^B))}{\text{Tr}(F_i^A N^B \rightarrow A (\sigma^B))} \]
The exact form of time-reversal is not implicit in the formalism!
Time reversal symmetry

The exact form of time-reversal is not implicit in the formalism!

The time-reversed image \( \{F_i\} \) of \( \{\rho_i\} \) is determined relative to preparations \( \{\sigma_m\} \) that have not been time-reversed.
Generalized Wigner’s theorem

**Important:** states and effects are objects that live in *different* spaces.

There is *no natural isomorphism* between the two spaces!

We represent them by operators in the same space based on the bilinear form

\[
(E^A^*, \rho^A) = \langle \rho^A, E^A \rangle = \text{Tr}[\rho^A E^A],
\]

which defines an isomorphism \( E^A^* \leftrightarrow E^A \).

*This isomorphism has no physical meaning!* It is simply based on the choice of bilinear form, and should not be confused with time reversal!
Generalized Wigner’s theorem

Two types of symmetry transformation:

**Type I** - States go to states, and effects go to effects: \((\hat{S}_{s\rightarrow s}^A, \hat{S}_{e\rightarrow e}^A)\)

**Type II** - States go to effects, and effects go to states: \((\hat{S}_{s\rightarrow e}^A, \hat{S}_{e\rightarrow s}^A)\)
**Generalized Wigner’s theorem**

- Symmetries of type I are described by:

\[
\hat{S}_{s\rightarrow s}(\rho; \bar{\rho}) = (\sigma; \bar{\sigma}) = \left( \frac{S\rho S^\dagger}{\text{Tr}(S\bar{\rho}S^\dagger)} ; \frac{S\bar{\rho}S^\dagger}{\text{Tr}(S\bar{\rho}S^\dagger)} \right),
\]

\[
\hat{S}_{e\rightarrow e}(E; \bar{E}) = (F; \bar{F}) = (d \frac{S^{-1}\dagger E S^{-1}}{\text{Tr}(S^{-1}\dagger \bar{E} S^{-1})} ; d \frac{S^{-1}\dagger \bar{E} S^{-1}}{\text{Tr}(S^{-1}\dagger \bar{E} S^{-1})}),
\]

or

\[
\hat{S}_{s\rightarrow s}(\rho; \bar{\rho}) = (\sigma; \bar{\sigma}) = \left( \frac{S\rho^T S^\dagger}{\text{Tr}(S\bar{\rho}^T S^\dagger)} ; \frac{S\bar{\rho}^T S^\dagger}{\text{Tr}(S\bar{\rho}^T S^\dagger)} \right),
\]

\[
\hat{S}_{e\rightarrow e}(E; \bar{E}) = (F; \bar{F}) = (d \frac{S^{-1}\dagger E^T S^{-1}}{\text{Tr}(S^{-1}\dagger \bar{E}^T S^{-1})} ; d \frac{S^{-1}\dagger \bar{E}^T S^{-1}}{\text{Tr}(S^{-1}\dagger \bar{E}^T S^{-1})}),
\]

where \( S \) is an invertible operator, and \( T \) is a transposition is some basis.
Generalized Wigner’s theorem

- Symmetries of type II are described by:

\[
\hat{S}_{s\rightarrow e}(\rho; \bar{\rho}) = (F; \bar{F}) = (d \frac{S\rho S^\dagger}{\text{Tr}(S\bar{\rho}S^\dagger)}; d \frac{S\bar{\rho}S^\dagger}{\text{Tr}(S\bar{\rho}S^\dagger)}),
\]

\[
\hat{S}_{e\rightarrow s}(E; \bar{E}) = (\sigma; \bar{\sigma}) = (\frac{S^{-1\dagger}}{\text{Tr}(S^{-1\dagger}E S^{-1})} \frac{E S^{-1}}{\text{Tr}(S^{-1\dagger}E S^{-1})}; \frac{S^{-1\dagger}}{\text{Tr}(S^{-1\dagger}E S^{-1})} \frac{\bar{E} S^{-1}}{\text{Tr}(S^{-1\dagger}\bar{E} S^{-1})}),
\]

or

\[
\hat{S}_{s\rightarrow e}(\rho; \bar{\rho}) = (F; \bar{F}) = (d \frac{S\rho^T S^\dagger}{\text{Tr}(S\bar{\rho}^T S^\dagger)}; d \frac{S\bar{\rho}^T S^\dagger}{\text{Tr}(S\bar{\rho}^T S^\dagger)}),
\]

\[
\hat{S}_{e\rightarrow s}(E; \bar{E}) = (\sigma; \bar{\sigma}) = (\frac{S^{-1\dagger}}{\text{Tr}(S^{-1\dagger}E^T S^{-1})} \frac{E S^{-1}}{\text{Tr}(S^{-1\dagger}E^T S^{-1})}; \frac{S^{-1\dagger}}{\text{Tr}(S^{-1\dagger}E^T S^{-1})} \frac{\bar{E} S^{-1}}{\text{Tr}(S^{-1\dagger}\bar{E}^T S^{-1})}).
\]

where \( S \) is an invertible operator, and \( T \) is a transposition is some basis.
If the evolution under time reversal is described by Schrödinger’s equation, positivity of energy → **time reversal is in the class:**

\[
\hat{S}_{s\rightarrow e}(\rho; \overline{\rho}) = (F; \overline{F}) = \left( d \frac{S \rho^T S^\dagger}{\text{Tr}(S \rho^T S^\dagger)} ; d \frac{S \overline{\rho}^T S^\dagger}{\text{Tr}(S \overline{\rho}^T S^\dagger)} \right),
\]

\[
\hat{S}_{e\rightarrow s}(E; \overline{E}) = (\sigma; \overline{\sigma}) = \left( \frac{S^{-1 \dagger} E S^{-1}}{\text{Tr}(S^{-1 \dagger} E^T S^{-1})} ; \frac{S^{-1 \dagger} \overline{E} S^{-1}}{\text{Tr}(S^{-1 \dagger} \overline{E}^T S^{-1})} \right).
\]

The standard notion corresponds to unitary \( S \).
Understanding the observed asymmetry


For an observer at $t_1$, all future circuits contain standard operations iff $\sum_{j \in Q} E_j = 1$.

(linked to the fact that we can remember the past and not the future)
Note: it is logically possible that non-standard operations were obtainable without post-selection
A time-neutral formalism

An isomorphism dependent on time reversal

\[
(\mathcal{M}^{A_1 \rightarrow B_1} ; \mathcal{\overline{M}}^{A_1 \rightarrow B_1} ) \leftrightarrow (\mathcal{M}^{A_1 B_2} ; \mathcal{\overline{M}}^{A_1 B_2} )
\]

O.O. and N. Cerf, arXiv: 1406.3829
A time-neutral formalism

Example:

\[ p(i, j) = \frac{\text{Tr}(\rho_i^A E_j^A)}{\text{Tr}(\bar{\rho}^A \bar{E}^A)} \]

\[ p(i, j) = \frac{\text{Tr}[(F_i^{A_2} \otimes E_j^{A_1}) |\Phi\rangle\langle\Phi|^{A_2A_1}]}{\text{Tr}[(F_i^{A_2} \otimes E_j^{A_1}) |\Phi\rangle\langle\Phi|^{A_2A_1}]} \]

\( F_i \) is the time-reversed image of \( \rho_i \).
A time-neutral formalism

Example:

\[ p(i, j) = \frac{Tr(\rho_i^A E_j^A)}{Tr(\bar{\rho}^A \bar{E}^A)} \]

\( \{E_j^A\} \) \( \{\rho_i^A\} \) \( \{F_i^{A_2}\} \)

entangled state

\[ |\Phi\rangle\langle\Phi|^{A_2 A_1} \]

the usual states and effects live on systems of type 1

\( F_i \) is the time-reversed image of \( \rho_i \).
A time-neutral formalism

Example:

\[
p(i, j) = \frac{\text{Tr}(\rho_i^A E_j^A)}{\text{Tr}(\rho^A \overline{E^A})}
\]

\[
\{ E_j^A \}
\]

\[
\{ \rho_i^A \}
\]

\[
\{ \sigma_j^{A_2} \}
\]

\[
\{ E_j^{A_1} \}
\]

\[
|\Phi\rangle\langle\Phi|^{A_2A_1}
\]

entangled state

\[
\text{entangled state}
\]

\[
\sigma_j
\]

is the time-reversed image of \( E_j \).

the time-reversed states and effects live on systems of type 2
A time-neutral formalism

Example:

\[ p(i, j) = \frac{\text{Tr}(\rho_i^A E_j^A)}{\text{Tr}(\bar{\rho}^A \bar{E}^A)} \]

\[ p(i, j) = \frac{\text{Tr}[(F_{i}^{A_2} \otimes E_{j}^{A_1})|\Phi\rangle\langle\Phi|^{A_2 A_1}]}{\text{Tr}[(\bar{F}^{A_2} \otimes \bar{E}^{A_1})|\Phi\rangle\langle\Phi|^{A_2 A_1}]} \]

\( F_i \) is the time-reversed image of \( \rho_i \).
A time-neutral formalism

An isomorphism dependent on time reversal

TRANSFORMATIONS

\[
(M^{A_1 \rightarrow B_1}; M^{A_1 \rightarrow B_1}^-) \leftrightarrow (M^{A_1 B_2}; M^{A_1 B_2}^-)
\]

EFFECTS ON PAIRS OF SYSTEMS

Joint probabilities:

\[
p(i,j,k,l \mid \{M_i^{A_2 B_2}\}, \{N_j^{C_2}\}, \ldots, W) = \\
\frac{\text{Tr}[W^{A_i A_2 B_2 B_1 C_1 C_2 D_1 D_2} (M_i^{A_2 B_2} \otimes N_j^{C_2} \otimes P_k^{B_2 C_2 D_2} \otimes Q_l^{A_1 D_1})]}{\sum_{i,j,k,l} \text{Tr}[W^{A_i A_2 B_1 B_2 C_1 C_2 D_1 D_2} (M_i^{A_2 B_2} \otimes N_j^{C_2} \otimes P_k^{B_2 C_2 D_2} \otimes Q_l^{A_1 D_1})]}
\]

= \langle \Phi | \Phi \rangle_{\Phi^{A_2 A_1}} \otimes \langle \Phi | \Phi \rangle_{\Phi^{B_2 B_1}} \otimes \langle \Phi | \Phi \rangle_{\Phi^{C_2 C_1}} \otimes \langle \Phi | \Phi \rangle_{\Phi^{D_1 D_2}}

'process matrix' (encodes the connections)
A time-neutral formalism

Can describe circuits with cycles:

\[ \{M_i\} \rightarrow \{N_j\} \rightarrow \{L_k\} \]

All such circuits can be realized using post-selection.

Compatible with closed timelike curves (P-CTC):

A time-neutral formalism

There exist circuits with cycles that can be obtained without post-selection!

The idea of background independence extended to random events

(provides a basis for understanding experiments with the quantum switch)
Equivalently:

\[
p(i, j, \cdots | \{M_i^{A_i A_2}\}, \{M_j^{B_i B_2}\}, \cdots, W) = \frac{\text{Tr}[W^{A_i A_2 B_i B_2 \cdots} (M_i^{A_i A_2} \otimes M_j^{B_i B_2} \otimes \cdots)]}{\text{Tr}[W^{A_i A_2 B_i B_2 \cdots} (\overline{M}^{A_i A_2} \otimes \overline{M}^{B_i B_2} \otimes \cdots)]}
\]

The 'process matrix':

\[
W^{A_i A_2 B_i B_2 \cdots} \geq 0, \quad \text{Tr}(W^{A_i A_2 B_i B_2 \cdots}) = 1
\]

Note: Any process matrix is allowed.

O.O. and N. Cerf, arXiv: 1406.3829
Time-symmetric process matrix formalism

Equivalently:

\[
p(i, j, \cdots | \{M_i^{A_1A_2}\}, \{M_j^{B_1B_2}\}, \cdots, W) = \frac{\text{Tr}[W^{A_1A_2B_1B_2}\cdots (M_i^{A_1A_2} \otimes M_j^{B_1B_2} \otimes \cdots)]}{\text{Tr}[W^{A_1A_2B_1B_2}\cdots (\bar{M}^{A_1A_2} \otimes \bar{M}^{B_1B_2} \otimes \cdots)]}
\]

Linked to two-time and multi-time state vector formalism:

Aharonov, Bergmann, Lebowitz, PRB 134, 1410 (1964)
Observation: The predictions are the same whether the systems are of type 1 or type 2.

Proposal: There is no a priori distinction between systems of type 1 and 2. The concept of time should come out from properties of the dynamics!
Dropping the assumption of local time

**Observation:** The predictions are the same whether the systems are of type 1 or type 2.

**Proposal:** There is no a priori distinction between systems of type 1 and 2.

The concept of time should come out from properties of the dynamics!

**The general picture:**

\[
p(i, j, \cdots | \{M_i\}, \{N_j\}, \cdots) = \frac{Tr[W_{\text{wires}}(M_i \otimes N_j \otimes \cdots)]}{Tr[W_{\text{wires}}(M_0 \otimes N_0 \otimes \cdots)]}
\]

Main probability rule

O.O. and N. Cerf, arXiv: 1406.3829
• Connecting operations amounts to new operations.

\[ L_{ij}^{abefg} = \frac{Tr_{cd}[|\Phi\rangle\langle\Phi|^c (M_{i}^{abc} \otimes N_{j}^{defg})]}{Tr[|\Phi\rangle\langle\Phi|^c (\bar{M}_{i}^{abc} \otimes \bar{N}_{j}^{defg})]} \]

(In some cases this may be the \textit{null} operation.)

• Every region performs a ‘measurement’ on the state prepared by its complement.

• There is an update rule for states and operations upon learning of information (not shown here).
Limit of quantum field theory


(the ‘general boundary’ approach with a few generalization)

O.O. and N. Cerf, arXiv: 1406.3829
Proposal: causal structure from correlations

The causal structure underlying the dynamics in the region is reflected in correlation properties of the state on the boundary.

O.O. and N. Cerf, arXiv: 1406.3829
Conclusion on the last part

It is possible to formulate a QT without any predefined time, which

- agrees with experiment
- has a physical and informational interpretation
- opens up the possibility to understand time and causal structure as dynamical and explore new forms of dynamics

• Is the metric/causal structure emergent, or do we need to postulate it as another field?

• What processes/networks can be realized without post-selection (e.g., can we violate causal inequalities?)

• How can we formulate general covariant laws of dynamics in this framework?

• What does it imply for the foundations of information processing?