A topological perspective on interacting algebraic theories

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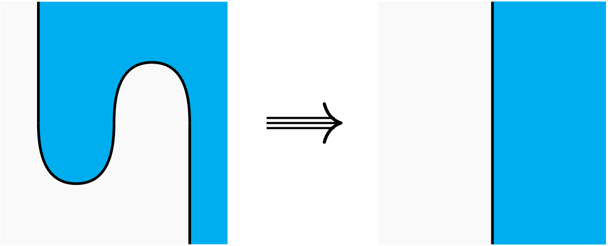
Glasgow, 9 June 2016

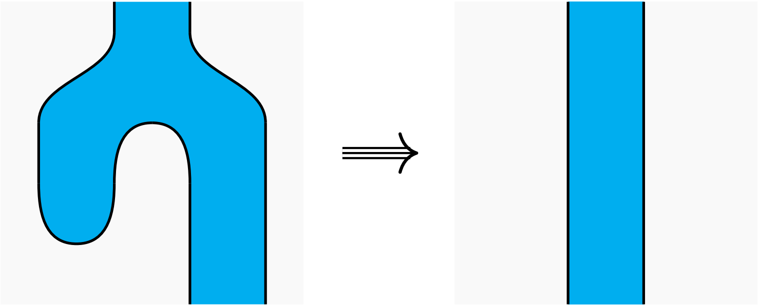
Convincing uses of string diagrams

Adjunctions: ✓✓



Convincing uses of string diagrams

Monoids (and **Frobenius algebras**): ✓✓



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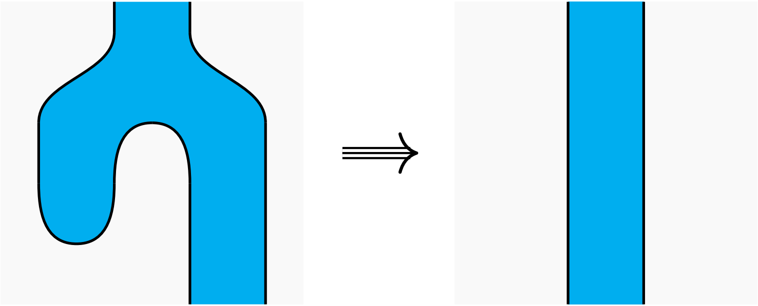
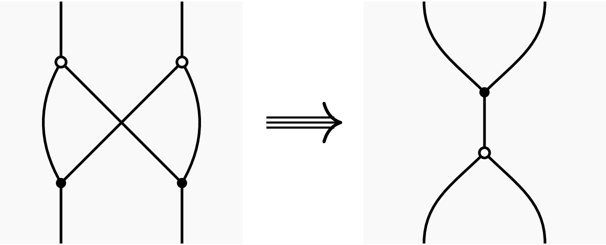


$\rightsquigarrow$ Spider laws in the ZX calculus

Convincing uses of string diagrams

Bialgebras: not really!



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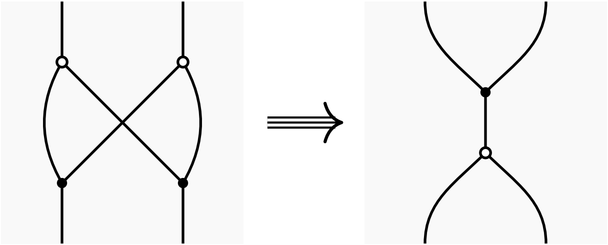
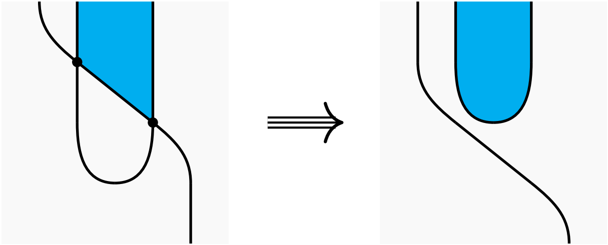


⇝ Find a topological explanation for this and similar laws

Convincing uses of string diagrams

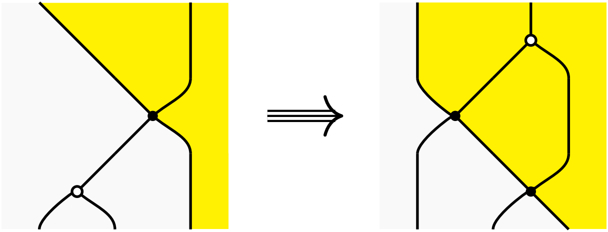
Teleportation (biunitaries): ✓



(Vicary, 2012, *Higher quantum theory*)

Convincing uses of string diagrams

Homomorphisms of monads and other naturality equations: ✓



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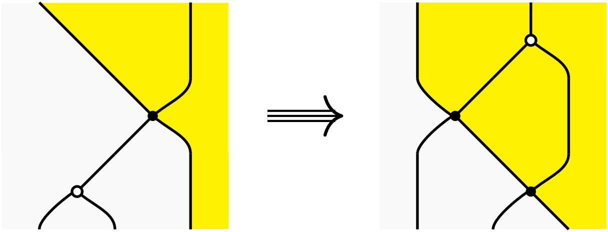

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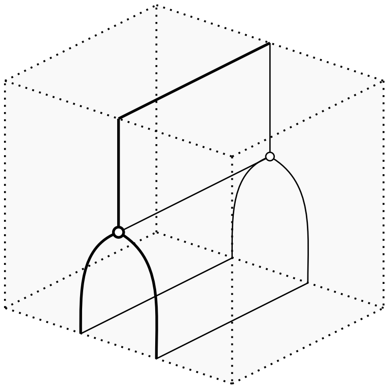
- Why does **“sliding”** appear in naturality equations?

A different angle



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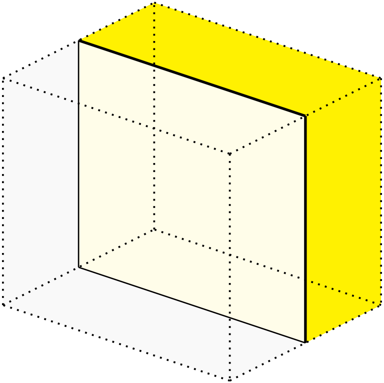

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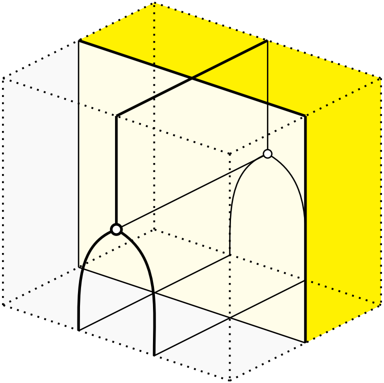

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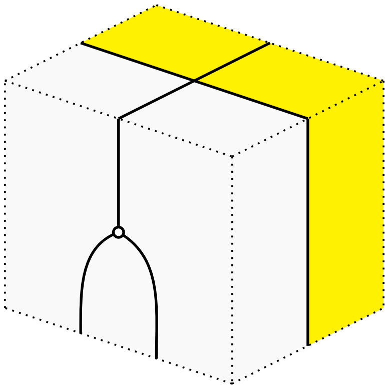

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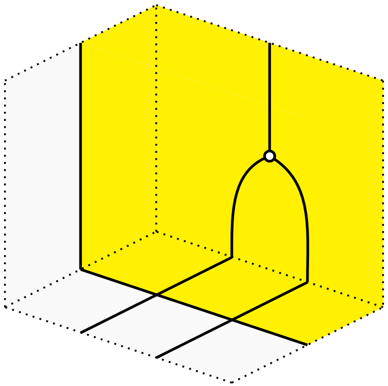

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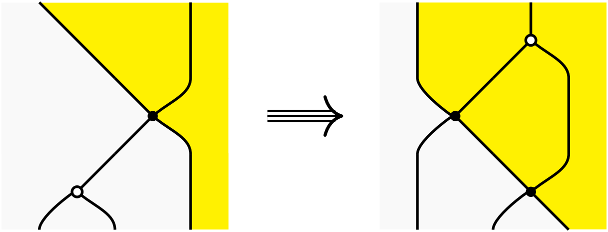

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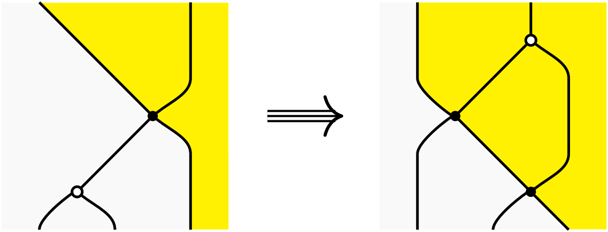

$\rightsquigarrow$ **Orthogonality** as the geometric correlate of **naturality**

Higher categories and directed spaces

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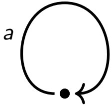

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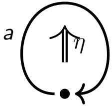

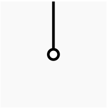

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(Also: disjoint unions, quotients, ...)

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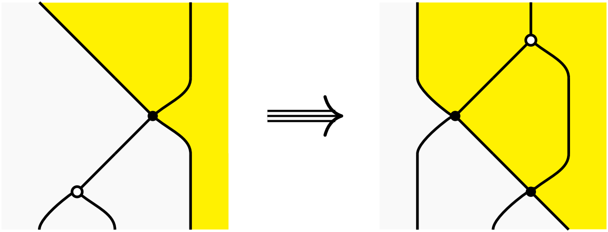

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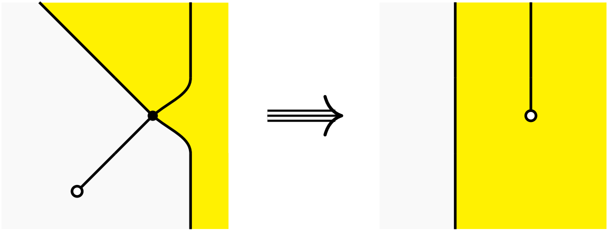

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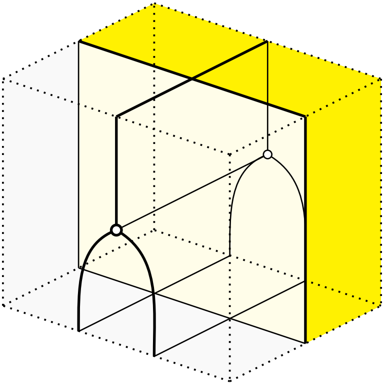

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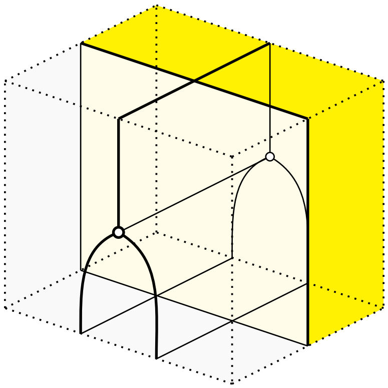

(In topology: **homotopy** of maps)

Topology of bialgebras

"Monoidal" theories (only 1 colour) are naturally **pointed spaces**

- Smash product $X \wedge Y$: quotient out $X \otimes \{*_Y\}$ and $\{*_X\} \otimes Y$ in $X \otimes Y$

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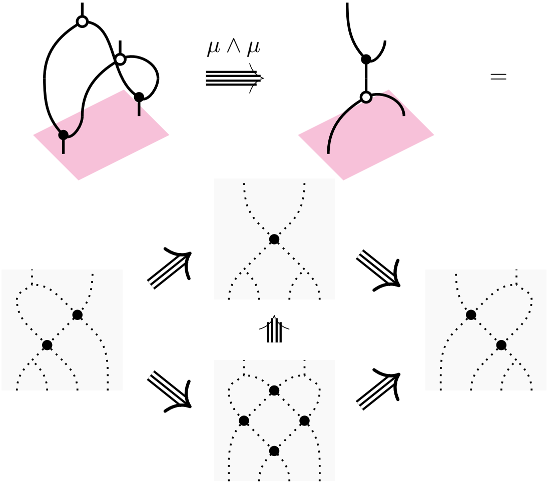
We consider the smash product $M \wedge M$.

Topology of bialgebras



Next

- A compositional approach to higher algebraic theories, importing tools from algebraic topology

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Thank you for your attention.