Operational Theories of Physics as Categories

Sean Tull

University of Oxford

sean.tull@cs.ox.ac.uk

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The Plan

Find connections between:

- General probabilistic theories
- Categorical approaches, particularly effectus theory.

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Tests, classical data	Coproducts
Causality	Discard maps / Terminal object

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Establish correspondence between operational categories and theories.

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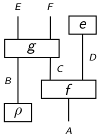

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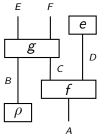

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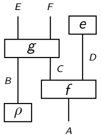

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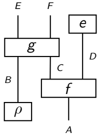

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  Call subsets $\{f_y\}_{y \in Y} \subseteq \{f_x\}_{x \in X}$ *partial tests*.
- Coarse-graining: Partial 'addition' $f \ovee g \colon A \to B$ on events.
  $\{f_x \colon A \to B\}_{x \in X} \cup \{g_y\}_{y \in Y} \implies \{\bigovee_{x \in X} f_x\} \cup \{g_y\}_{y \in Y}$

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Examples

Many! Classical: deterministic or probabilistic.
Quantum: Hilbert spaces or C*-algebras and c.p. sub-unital maps.

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Encode outcome sets X 'in the systems'.

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Categorically, direct sums are finite coproducts $(+, 0)$:

$$B_i \xrightarrow{\kappa_i} B_1 + \ldots + B_n = \bigoplus_{k=1}^{n} B_k \xrightarrow{\triangleright_j} B_j \ , \quad \triangleright_j \circ \kappa_i = \begin{cases} \mathrm{id} & i = j \\ 0 & i \neq j \end{cases}$$

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- $\otimes$ distributes over $+$
- $\stackrel{\cdot}{=}_{A+B} = [\stackrel{\cdot}{=}_A, \stackrel{\cdot}{=}_B]$, $\stackrel{\cdot}{=}_I = \mathrm{id}$, $\stackrel{\cdot}{=}_{A \otimes B} = \lambda \circ (\stackrel{\cdot}{=}_A \otimes \stackrel{\cdot}{=}_B)$

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Weakening of notion of monoidal effectus (Jacobs et al.).

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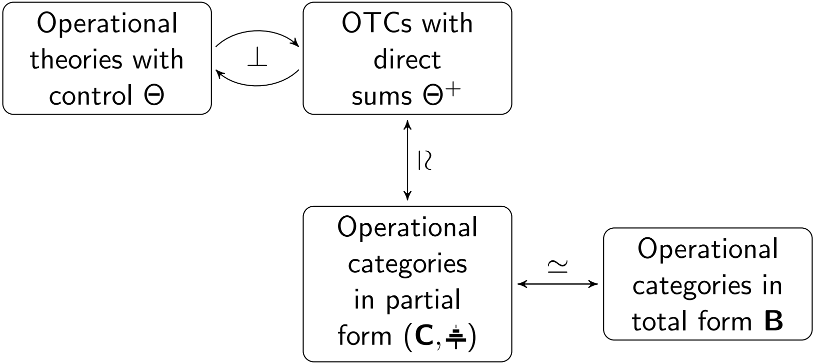
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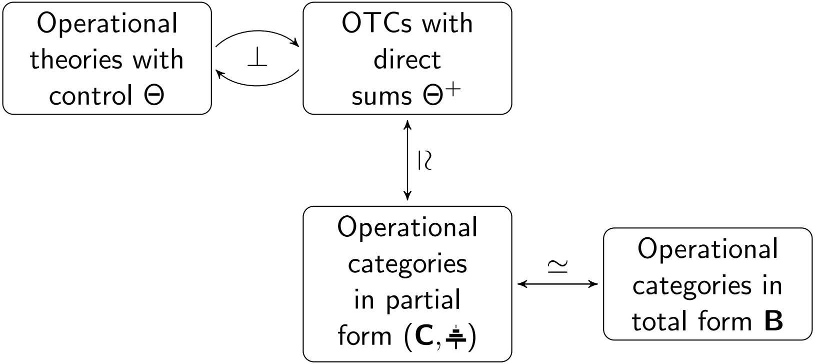

Examples

Θ	$\mathbf{B}$	$\mathbf{C}$
Classical	**Set**	**PFun**
Quantum	$\mathbf{CStar}^{\mathrm{op}}_{\mathrm{cpu}}$	$\mathbf{CStar}^{\mathrm{op}}_{\mathrm{cpsu}}$

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