Orthogonal Quantum Latin Squares

and Mutually Unbiased Bases

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Latin square

Definition

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$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$
$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$

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Let $\mathcal{P} :=$ , $\mathcal{Q} :=$  be a pair of orthogonal quantum Latin squares,  be the computational basis spider and H_j and G_q be indexed families of Hadamard matrices.

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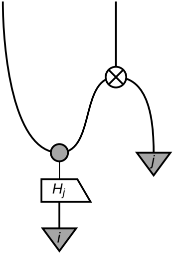
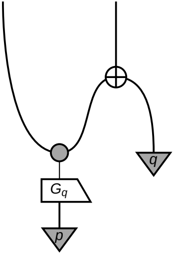
$A_{ij} := \frac{1}{\sqrt{n}}$ 

$B_{pq} := \frac{1}{\sqrt{n}}$ 

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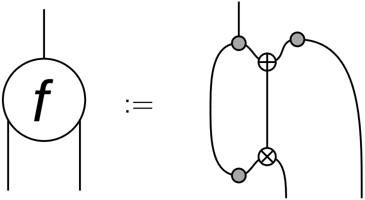
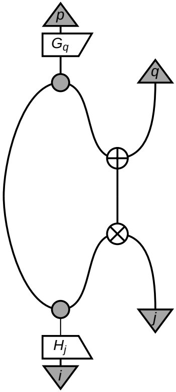


Proof

$$|\langle B_{pq}|A_{ij}\rangle|^2 \quad = \quad \left| \frac{1}{n} \quad \cdots \quad \right|^2$$



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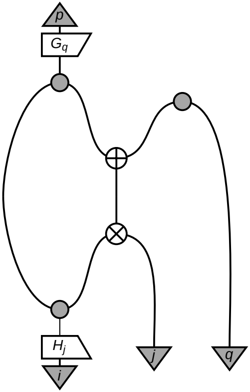

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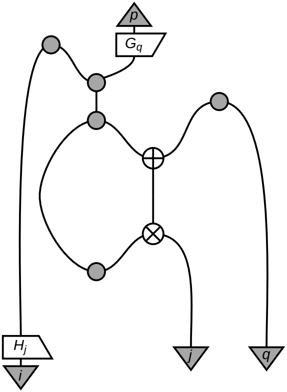
$$= \frac{1}{n^2} \left| \vphantom{\int} \right.$$  $$\left. \vphantom{\int} \right|^2$$

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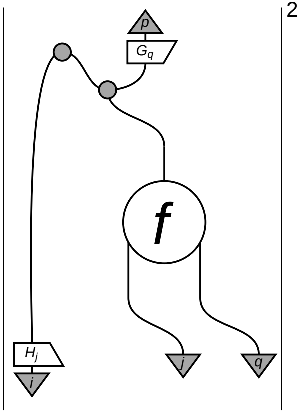

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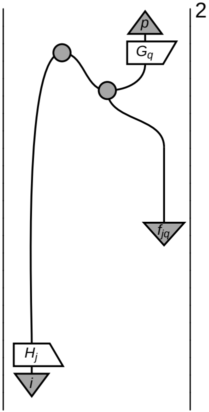

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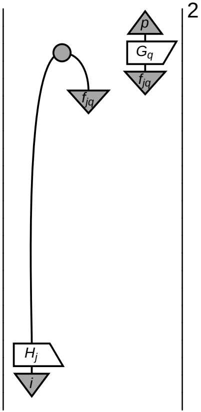

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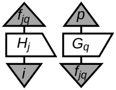
$= \frac{1}{n^2} \Bigg|$  $\Bigg|^2$

□

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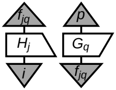
$$= \frac{1}{n^2} \left| \right.$$  $$\left. \right|^2$$

$$= \frac{1}{n^2} \left| (H_j)_{it} (G_q^\dagger)_{tp} \right|^2$$

$\square$

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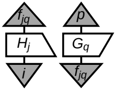
$$= \frac{1}{n^2} \left| \right.$$  $$\left. \right|^2$$

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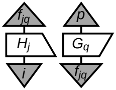

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