On the Cohomology of Contextuality

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Outline

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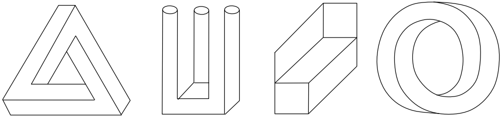
2. Contextuality and "impossible figures" in Topology.



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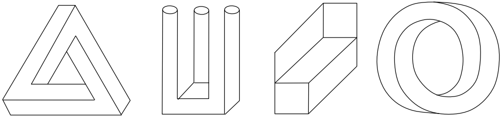

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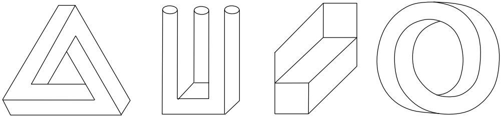

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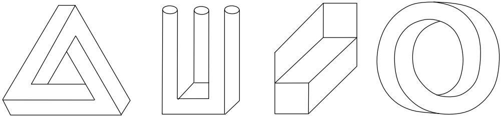

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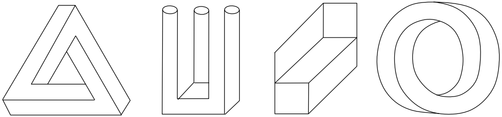

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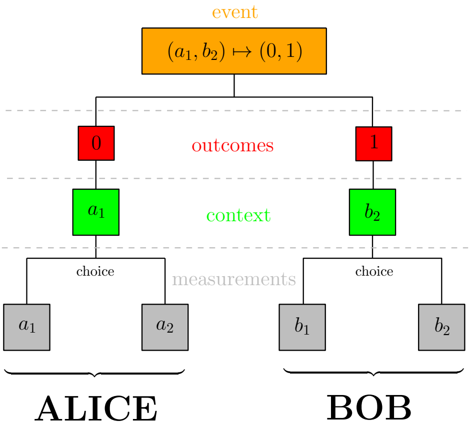
Basic scenario: Two agents Alice and Bob choose between two binary measurements each, in a $(2,2,2)$ Bell-type scenario:



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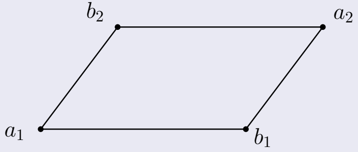
Example: $(2, 2, 2)$ Bell-type scenario



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A	B	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
a_1	b_1	$1/2$	0	0	$1/2$
a_2	b_1	$3/8$	$1/8$	$1/8$	$3/8$
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The support of a probabilistic empirical model determines a **possibilistic empirical model**.

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3. Every family $\{s_C \in \mathcal{S}(C)\}_{C \in \mathcal{M}}$ which is compatible (i.e. such that $s_C \mid_{C \cap C'} = s_{C'} \mid_{C \cap C'}$ for all $C, C' \in \mathcal{M}$) induces a unique global section in $\mathcal{S}(X)$.

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- $\mathcal{S}$ is **strongly contextual**, or $\mathrm{SC}(\mathcal{S})$, if $\mathrm{LC}(\mathcal{S}, s)$ for all s. In other words there is no global section ($\mathcal{S}(X) = \emptyset$).

Bundle diagrams

Bundle diagrams can be very helpful in representing models:

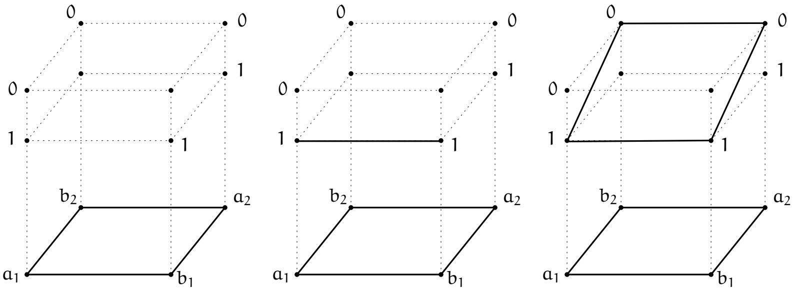

Figure: Left: a $(2,2,2)$ scenario. Centre: the section $(a_1, b_1) \mapsto (1,1)$. Right: the global section $(a_1, b_1, a_2, b_2) \mapsto (1,1,0,0)$

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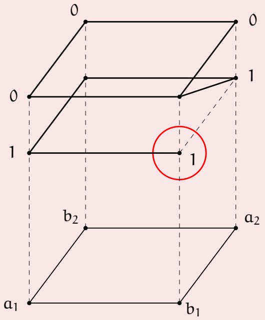

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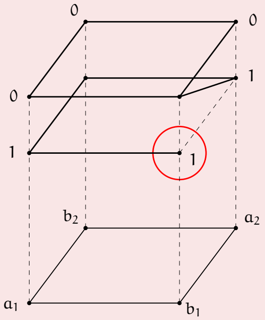


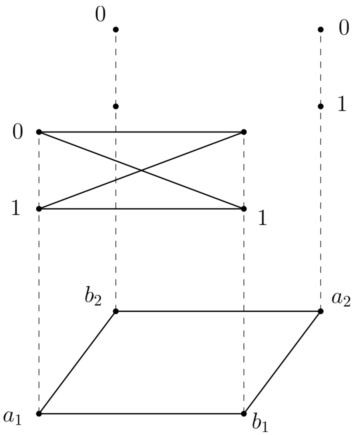
Here, the event $b_1 \mapsto 1$ **depends on the choice of Alice**. It is possible if Alice chooses a_1 yet impossible if she chooses a_2.

Examples

Hardy is logically contextual but not strongly contextual

Hardy model

A	B	00	10	01	11
a_1	b_1	1	1	1	1



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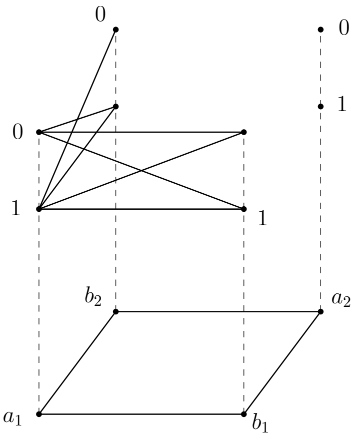

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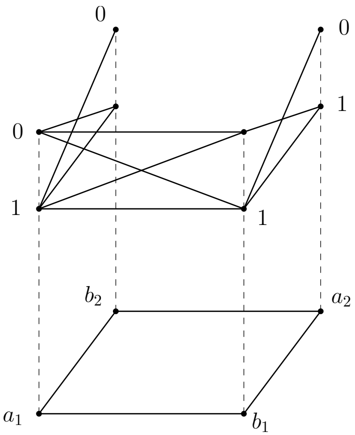

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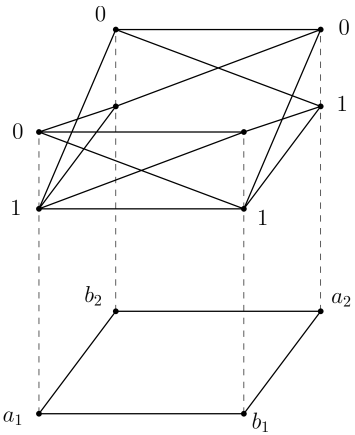

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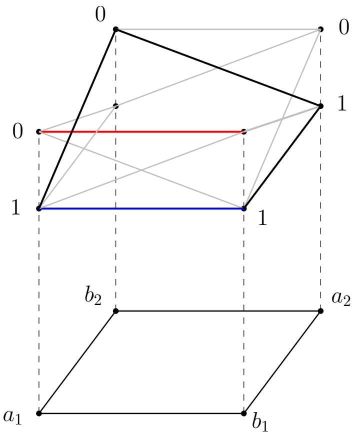
The red section is not contained in any compatible family

The blue section is contained in a compatible family



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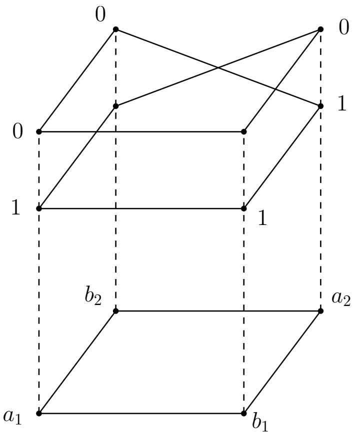

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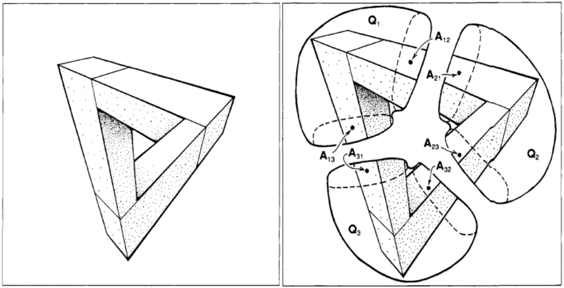

Figure: From R. Penrose's *On the Cohomology of Impossible Figures*

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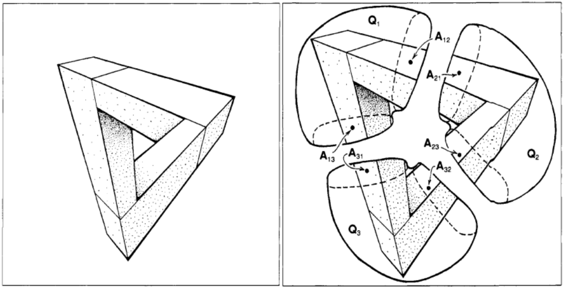

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Sheaf cohomology is used in Algebraic Geometry/Topology as a tool to study the extendability from local to global.

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- Can higher cohomology groups be used for the study of contextuality?
- Is there a concrete way of describing cohomological obstructions?

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- For each local section $s \in \mathcal{S}(C)$, associate a **cohomology obstruction** $\gamma_C(s) \in \check{H}^1(\mathcal{M}, \mathcal{F} \mid_C)$

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$$\mathcal{F} : \mathcal{P}(X)^{op} \to \mathbf{Set} \xrightarrow{F_{\mathbb{Z}}} \mathbf{AbGrp},$$

  which allows **formal linear combinations of local sections.**
- Define **Čech cohomology** $\check{H}^*(\mathcal{M}, \mathcal{F})$ of $\mathcal{F}$ w.r.t. the cover $\mathcal{M}$.
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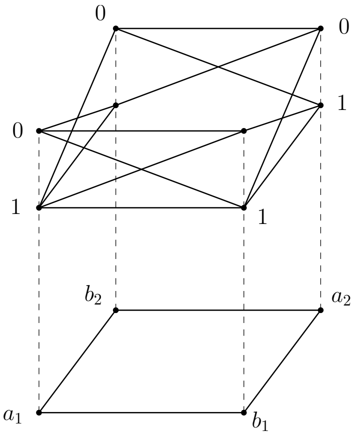
Theorem

$CLC(\mathcal{S}, s) \Rightarrow LC(\mathcal{S}, s)$, and $CSC(\mathcal{S}) \Rightarrow SC(\mathcal{S})$

False positives

Hardy model

A	B	00	10	01	11
a_1	b_1	1	1	1	1
a_1	b_2	0	1	1	1
a_2	b_1	0	1	1	1
a_2	b_2	1	1	1	0



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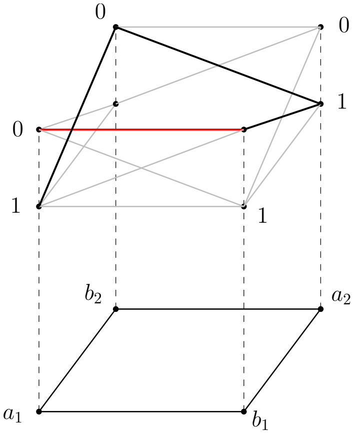

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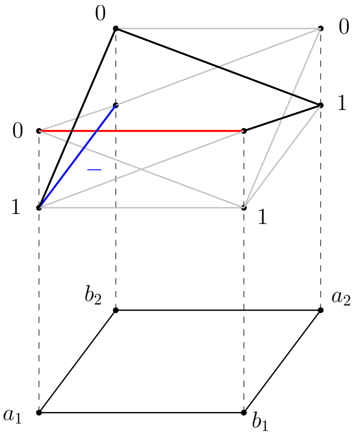

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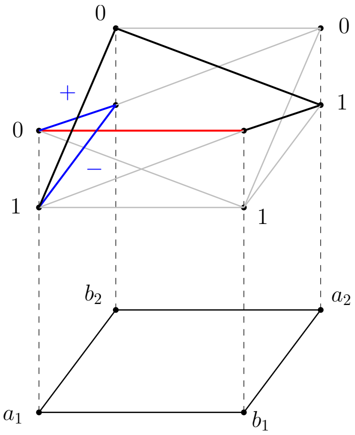
The possibility of **linearly adding sections** allows us to find a compatible family (for $\mathcal{F}$) containing the red section. Thus γ(red section) $= 0$, which is a **false positive!**



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The only known example of a strongly contextual false positive is the Kochen-Specker model for the cover

$$\{A, B, C\}, \{B, D, E\}, \{C, D, E\}, \{A, D, F\}, \{A, E, G\},$$

which "**does not satisfy any reasonable criterion for symmetry, nor does it satisfy any strong form of connectedness**" and where "**the existence of measurements belonging to a single context [...] seems to be crucial**" [Abramsky et al. QPL 2011].

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Under suitable assumptions of symmetry and connectedness of the cover, the cohomology obstruction is a complete invariant for strong contextuality.

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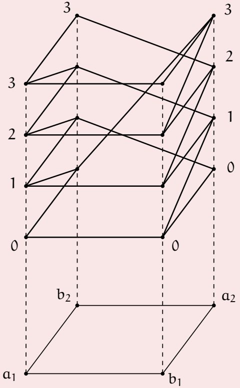
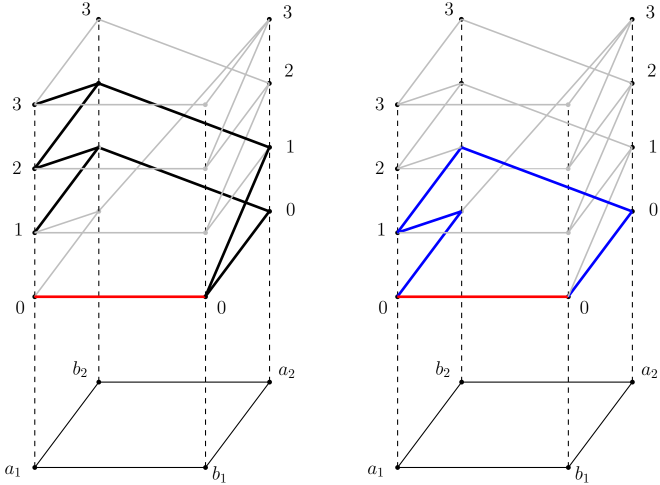
Counterexample on a $(2, 2, 4)$ Bell-type scenario



Counterexample



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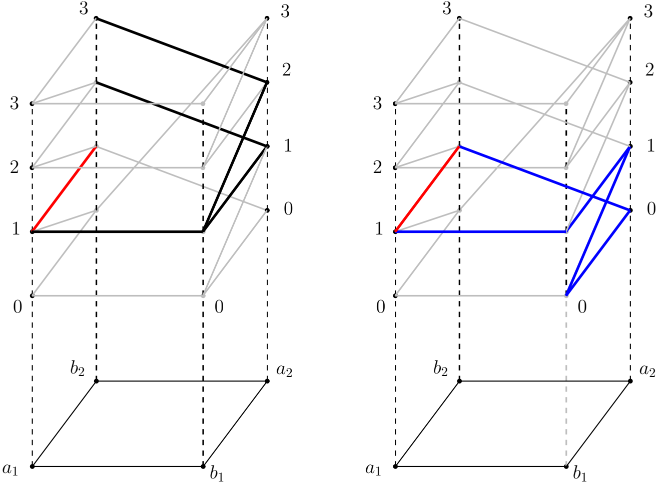

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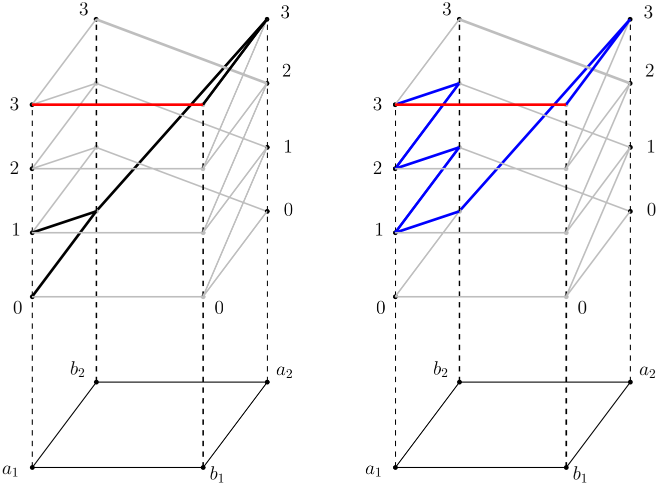

Higher cohomology groups

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- These "levels" are organised in a **hierarchy of logical implications.**
- The hierarchy cannot be used to study no-signalling empirical models.

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The first step consists of turning local sections at a context $C \in \mathcal{M}$ into relative cocycles, using the isomorphism $\psi^0_C : \mathcal{F}(C) \to Z^0(\mathcal{M}, \mathcal{F}\mid_C)$:

$$\mathcal{S}(C) \hookrightarrow \mathcal{F}(C) \xrightarrow{\cong} Z^0(\mathcal{M}, \mathcal{F}\mid_C) \cong \check{H}^1(\mathcal{M}, \mathcal{F}\mid_C) \xrightarrow{\gamma_C} \check{H}^1(\mathcal{M}, \mathcal{F}_{\tilde{C}}).$$

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Then, the cohomology obstruction $\gamma_C(s)$ is defined using the **connecting homomorphism** of the cohomology LES.

Higher cohomology groups

$$
\begin{array}{ccccccccc}
 & & & & & & \mathcal{F}(C) & & \\
 & & & & & & \downarrow{\scriptstyle \psi^0_C} & & \\
 & & Z^0(\mathcal{M},\mathcal{F}_{\tilde C}) & \longrightarrow & Z^0(\mathcal{M},\mathcal{F}) & \longrightarrow & Z^0(\mathcal{M},\mathcal{F}|_C) & & \gamma^0_C \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0(\mathcal{M},\mathcal{F}_{\tilde C}) & \longrightarrow & C^0(\mathcal{M},\mathcal{F}) & \longrightarrow & C^0(\mathcal{M},\mathcal{F}|_C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
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\end{array}
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It is possible to generalise the isomorphism

$$\psi_C^0 : \mathcal{F}(C) \xrightarrow{\cong} Z^0(\mathcal{M}, \mathcal{F}\mid_C) \xrightarrow{\text{incl.}} C^0(\mathcal{M}, \mathcal{F}\mid_C)$$

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\mathcal{F}(C) & \xhookrightarrow{\tilde{\psi}_C^q} & C^q(\mathcal{M}, \mathcal{F}|_C) \\
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However, its image is contained in $Z^q(\mathcal{M}, \mathcal{F})$ **only in even dimensions**. As a result, the obstruction is generalisable only in **odd-dimensional cohomology groups**.

Higher cohomology groups

$$
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 & & & & & & \mathcal{F}(C) & & \\
 & & & & & & \downarrow \psi^{2q}_C & & \\
 & & Z^{2q}(\mathcal{M},\mathcal{F}_{\tilde{C}}) & \longrightarrow & Z^{2q}(\mathcal{M},\mathcal{F}) & \longrightarrow & Z^{2q}(\mathcal{M},\mathcal{F}\,|_C) & \xrightarrow{\tilde{\gamma}^q_C} & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{2q}(\mathcal{M},\mathcal{F}_{\tilde{C}}) & \longrightarrow & C^{2q}(\mathcal{M},\mathcal{F}) & \longrightarrow & C^{2q}(\mathcal{M},\mathcal{F}\,|_C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^{2q+1}(\mathcal{M},\mathcal{F}_{\tilde{C}}) & \longrightarrow & Z^{2q+1}(\mathcal{M},\mathcal{F}) & \longrightarrow & Z^{2q+1}(\mathcal{M},\mathcal{F}\,|_C) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
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Thus we obtain a **refinement of the notion of cohomological contextuality**:

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Definition

*Let $s \in \mathcal{F}(C)$. We define the q-***th cohomological obstruction** *of s as the element*

$$\gamma_C^q(s) := \tilde{\gamma}_C^q(\psi^{2q}(s)) \in \check{H}^{2q+1}(\mathcal{M}, \mathcal{F}_{\tilde{C}}).$$

The empirical model $\mathcal{S}$ underlying $\mathcal{F}$ is defined to be

- **cohomologically logically** q**-contextual at a section** s, *or $CLC^q(\mathcal{S}, s)$, if $\gamma_C^q(s) \neq 0$. We say that $\mathcal{S}$ is* **cohomologically logically** q**-contextual** *if $CLC^q(\mathcal{S}, s)$ for some section s.*
- **cohomologically strongly** q**-contextual**, *or $CSC^q(\mathcal{S})$, if $CLC^q(\mathcal{S}, s)$ for all s.*

Higher cohomology groups

Theorem

These "levels" of contextuality are organised in the following hierarchy:

$$\begin{array}{ccccccccc}
CSC(\mathcal{S}) & \Longleftarrow & CSC^1(\mathcal{S}) & \Longleftarrow & \dots \Longleftarrow & CSC^q(\mathcal{S}) & \Longleftarrow & CSC^{q+1}(\mathcal{S}) & \Longleftarrow \dots \\
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However, **nothing is gained for no-signalling empirical models**:

Proposition

No-signalling empirical models are cohomologically q-non-contextual for any $q \gneq 0$.

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It remains an **open question** to identify possible applications of the hierarchy outside the framework of no-signalling models.

Connecting homomorphism and first cohomology group

Many contextual properties of a model can be inferred by the properties of the connecting homomorphism γ

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Proposition (Sufficient Condition for Strong Contextuality)

If there exists a $C \in \mathcal{M}$ such that γ_C is injective, then the empirical model is strongly contextual.

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The main ingredient are **torsors relative to an abelian presheaf**.

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Let $\mathcal{F} : \mathbf{Open}(X)^{op} \to \mathbf{AbGrp}$ be a presheaf of abelian groups over a topological space X.

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$$\phi_U : \mathcal{F}(U) \times T(U) \to T(U) :: (g, t) \mapsto g \bullet t$$

  is a left action of $\mathcal{F}(U)$ on $T(U)$.
- Given two $\mathcal{F}$-presheaves T and T', a **morphism of** $\mathcal{F}$**-presheaves** from T to T' is a morphism of presheaves $\psi : T \Rightarrow T'$ such that ψ_U is equivariant for all open $U \subseteq X$.
- An $\mathcal{F}$-presheaf T is called an $\mathcal{F}$-torsor if
  1. There exists an open cover $\mathcal{V}$ of X that **trivialises** T, i.e. such that $T(V) \neq \emptyset$ for all $V \in \mathcal{V}$.
  2. The action $\phi_U : \mathcal{F}(U) \times T(U) \to T(U)$ is simply transitive.

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Despite their seemingly sophisticated definition, **torsors are very simple objects**. Thus, this isomorphism allows us to **concretely understand cohomology obstructions**.

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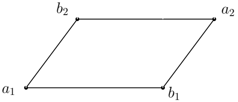
Σ



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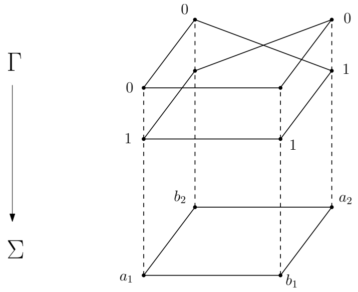

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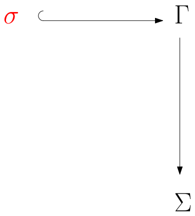

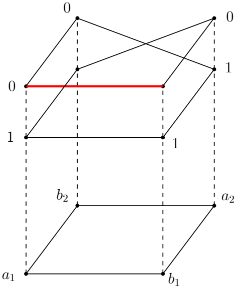

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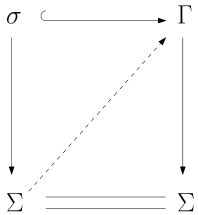

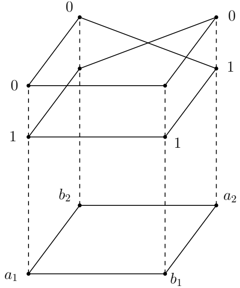

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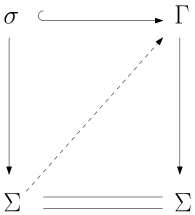

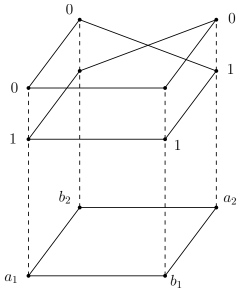

Relative extension problem from **obstruction theory**, which provides invariants to the extension of local maps in a cohomology with coefficients in the **homotopy groups.**

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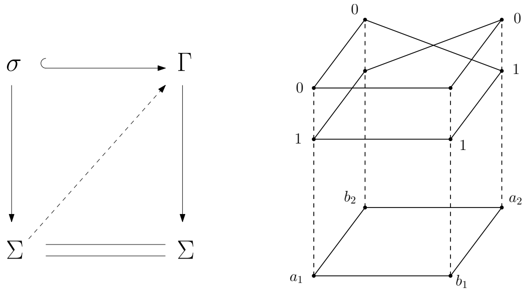

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The theory of **Postnikov towers** is central in this approach.