Cohomology of effect algebras

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Motivation: Bell's experiment

Alice and Bob select one of two binary measurements.
Alice's measurements: a, a' with possible outcomes $0, 1$
Bob's measurements: b, b' with possible outcomes $0, 1$
Probabilities of joint outcomes:

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
(a, b)	$1/2$	0	0	$1/2$
(a, b')	$3/8$	$1/8$	$1/8$	$3/8$
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- This setup is quantum mechanically realizable.
- It is not classically realizable.

Goal: develop systematic techniques to study realizability.

Motivation

- Bell-type scenarios can be described using topology of measurement covers. (Abramsky, Brandenburger)
- Using topological cohomology theory, one obtains a criterion for classical realizability. (Abramsky, Mansfield, Soares Barbosa)
- Bell-type scenarios can also be described using effect algebras. (Staton, Uijlen)
- Can we define cohomology of effect algebras?

Outline

1. Effect algebras
2. Cohomology
3. Applications to Bell-type scenarios

Effect algebras

Key feature of quantum logic: Partiality

$$A = \text{“The particle } P \text{ is at position } x_0\text{.”}$$
$$B = \text{“The particle } P \text{ has momentum } p_0\text{.”}$$

Conjunction $A \wedge B$ is not defined in this case

Effect algebras

Definition

An effect algebra consists of:

- A set A
- A partial binary operation $\oplus$ on A
- Constants $0, 1 \in A$
- An orthocomplement operation $(-)^{\perp} : A \to A$

such that

- The operation $\oplus$ is commutative and associative and has 0 as neutral element
- For every $a \in A$, $a^{\perp}$ is the unique element for which $a \oplus a^{\perp} = 1$
- $0^{\perp} = 1$
- If $a \oplus 1$ is defined, then $a = 0$

Effect algebras

Examples

- The unit interval $[0, 1]$ is an effect algebra, with addition as $\oplus$ and $a^\perp = 1 - a$.
- Let H be a Hilbert space. Then

$$\mathcal{E}f(H) = \{A : H \to H \mid 0 \leq A \leq I\}$$

  forms an effect algebra with the same operations.
- Any Boolean algebra is an effect algebra. $a \oplus b$ is defined whenever $a \wedge b = 0$, and in that case $a \oplus b = a \vee b$.
- Similarly, any orthomodular poset is an effect algebra.

States

Effects represent measurements on a physical system. To each effect algebra we associate a state space representing the corresponding states.

$$\mathrm{St}(A) = \left\{ \sigma : A \to [0,1] \;\middle|\; \begin{array}{l} \sigma(a \oplus b) = \sigma(a) + \sigma(b) \\ \sigma(1) = 1 \end{array} \right\}$$

State spaces

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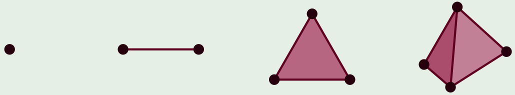
Example

The state space of $\mathcal{P}(n)$ consists of $a_1, \ldots, a_n \in [0,1]$ such that $a_1 + \cdots + a_n = 1$.

$\mathrm{St}(\mathcal{P}(1))$ $\quad$ $\mathrm{St}(\mathcal{P}(2))$ $\quad$ $\mathrm{St}(\mathcal{P}(3))$ $\quad$ $\mathrm{St}(\mathcal{P}(4))$



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Example

The state space of $\mathcal{E}f(H)$ is the set of density matrices on H, i.e. all positive $\rho : H \to H$ for which $\mathrm{tr}(\rho) = 1$.

State spaces

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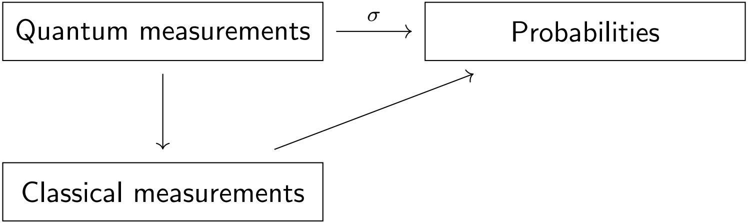
The state space always forms a compact convex space:
if σ, τ are states and $\lambda \in [0,1]$, then

$$\lambda\sigma + (1-\lambda)\tau$$

is again a state.

Bell's experiment using effect algebras



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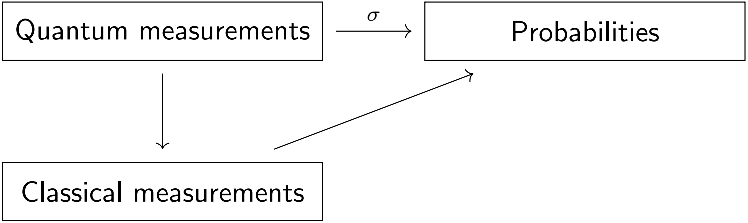

Measurements and probabilities can be modeled by effect algebras.

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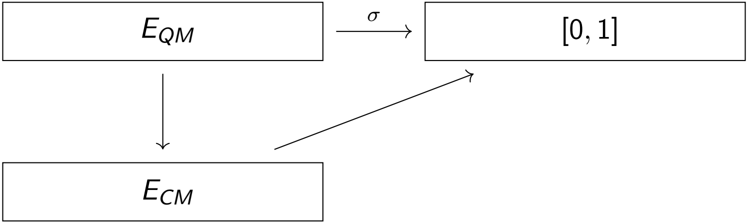

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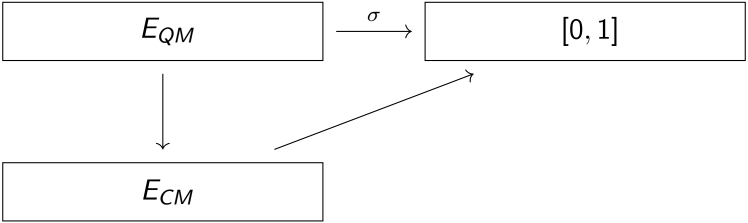


Measurements and probabilities can be modeled by effect algebras.

Theorem (Staton, Uijlen)

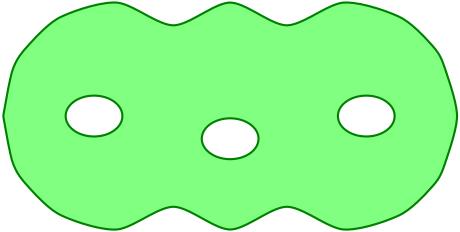
σ *classically realizable* $\iff$ σ *factors through* E_{CM}

Idea of cohomology

In topology:

$$\text{Functors } H^0, H^1, H^2, \ldots : \mathbf{TopSp} \to \mathbf{AbGrp}$$

$H^n(X)$ provides information about holes in the space X, or about extending loops to disks.



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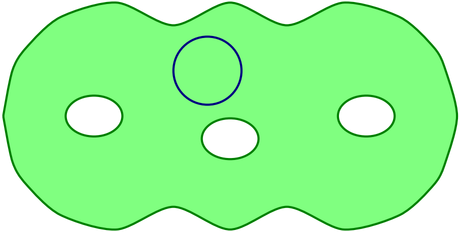

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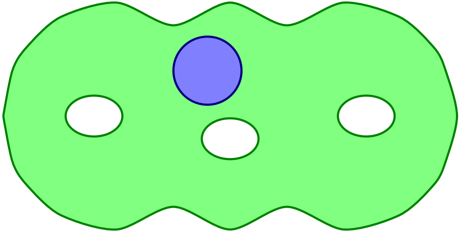

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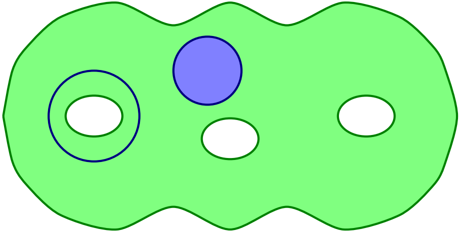

Idea of cohomology

In effect algebra theory:

$$\text{Functors } H^0, H^1, H^2, \ldots : \textbf{EffAlg} \to \textbf{AbGrp}$$

$H^n(A)$ provides information about states and state extensions.

$$\begin{array}{ccc} E_{QM} & \xrightarrow{\sigma} & [0,1] \\ \downarrow & \nearrow & \\ E_{CM} & & \end{array}$$

Constructing the cohomology groups

We modify Connes' definition of cyclic cohomology.

1. Tests on an effect algebra A:

$$T_n(A) = \{(a_0, \ldots, a_n) \mid a_0 \oplus \cdots \oplus a_n = 1\}$$

Constructing the cohomology groups

We modify Connes' definition of cyclic cohomology.

1. Tests on an effect algebra A:

$$T_n(A) = \{(a_0, \dots, a_n) \mid a_0 \oplus \cdots \oplus a_n = 1\}$$

2. Operations on tests:

$$\begin{aligned}
d_i :\quad & T_n(A) \to T_{n-1}(A) \\
d_0 : (a_0, \dots, a_n) & \mapsto (a_0 \oplus a_1, a_2, \dots, a_n) \\
d_1 : (a_0, \dots, a_n) & \mapsto (a_0, a_1 \oplus a_2, a_3, \dots, a_n) \\
& \vdots \\
d_n : (a_0, \dots, a_n) & \mapsto (a_n \oplus a_0, a_1, \dots, a_n)
\end{aligned}$$

Constructing the cohomology groups

$$T_n(A) = \{(a_0, \dots, a_n) \mid a_0 \oplus \cdots \oplus a_n = 1\}$$

3. Cocycles:

$$\mathcal{C}^n(A) = \left\{ \varphi : T_n(A) \to \mathbb{R} \;\middle|\; \begin{array}{l} \varphi(a_n, a_0, \dots, a_{n-1}) \\ = (-1)^n \varphi(a_0, \dots, a_n) \end{array} \right\}$$

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Example

For $n = 1$:

$$\varphi(a, b) = -\varphi(b, a)$$

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Example

For $n = 1$:

$$\varphi(a, a^\perp) = -\varphi(a^\perp, a)$$

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4. Operations on cocycles:

$$\begin{aligned} d^i &: \ \mathcal{C}^{n-1}(A) \to \mathcal{C}^n(A) \\ d^i\varphi &= \left(T_n(A) \xrightarrow{d_i} T_{n-1}(A) \xrightarrow{\varphi} \mathbb{R} \right) \end{aligned}$$

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5. Cochain complex:

$$\mathcal{C}^0(A) \begin{array}{c} -\, d^0 \to \\ -\, d^1 \to \end{array} \mathcal{C}^1(A) \begin{array}{c} -\, d^0 \to \\ -\, d^1 \to \\ -\, d^2 \to \end{array} \mathcal{C}^2(A) \begin{array}{c} -\, d^0 \to \\ -\, d^1 \to \\ -\, d^2 \to \\ -\, d^3 \to \end{array} \cdots$$

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$$\delta^n = d^0 - d^1 + d^2 - \cdots \pm d^n$$

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6. $H^n(A) = \ker(\delta^{n+1}) / \operatorname{im}(\delta^n)$

Cohomology

Examples

- Cohomology of the unit interval $[0, 1]$:

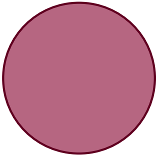
$$H^0([0,1]) = \mathbb{R}$$
$$H^n([0,1]) = 0 \quad \text{for } n > 0$$

- Cohomology of the Boolean algebra $\mathcal{P}(m)$:

$$H^n(\mathcal{P}(m)) = \mathbb{R}^{\binom{m-1}{n}}$$

First cohomology group

The first cohomology group is related to the state space.
The state space is always a compact convex space.
Every convex space can be embedded in an $\mathbb{R}$-vector space.
In fact, there is a smallest vector space in which it embeds:



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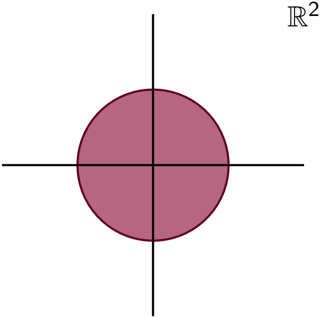
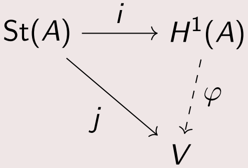


First cohomology group

Theorem

Let A be a finite effect algebra that has enough states. Then $H^1(A)$ is the smallest vector space in which $\mathrm{St}(A)$ can be embedded:



"A has enough states" means:
if $\sigma(a) = \sigma(b)$ for all states σ, then $a = b$.

Applications

Let $\sigma : A \to [0,1]$ be a state on an effect algebra A, for instance the Bell state. Then:

$$\sigma \text{ classically realizable} \iff \begin{array}{c}\sigma \text{ factors through} \\ \text{a Boolean algebra } B\end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & [0,1] \\ \downarrow & \nearrow & \\ B & & \end{array}$$

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$$\begin{array}{ccccc} \mathrm{St}(A) & \xrightarrow{\ i\ } & H^1(A) & \xrightarrow{\ \partial\ } & H^2(B,A) \\ \sigma & \longmapsto & & & \partial(i(\sigma)) \end{array}$$

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$$\Downarrow \quad \not\Uparrow$$

$$\partial(i(\sigma)) = 0$$

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How to get rid of false positives?
Adapt the definition of cohomology!

$$\mathcal{C}^n_{\leq}(A) = \left\{ \left(\begin{array}{l} \varphi : T_n(A) \to \mathbb{R}, \\ \psi : T_{n-1}(A) \to \mathbb{R} \end{array} \right) \middle| \; \varphi \geq \delta\psi \right\}$$

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σ classically realizable $\Longleftrightarrow$ σ factors through a Boolean algebra B

$\Updownarrow$

$\partial(i(\sigma)) \geq 0$

Conclusions

- Effect algebras can be used to model contextuality scenarios.
- Cohomology of effect algebras is relatively easy to compute, and contains information about states and classical realizability.
- Order cohomology provides a criterion for classical realizability without false positives, but is more difficult to compute.