(Modular) effect algebras are equivalent to (Frobenius) antispecial algebras

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QPL 2016
Glasgow, June 2016
Effect algebra

Definition

An *effect algebra* is a set $A$ together with the signature of partial functions

$$A \times A \rightarrow A \leftarrow A \leftrightarrow I$$

where

- $(A, \otimes, 0)$ is a commutative monoid,
- the following conditions are satisfied for all $x, y \in A$

$$x \otimes y = 1 \iff x = \neg y$$
$$x \otimes 1 = 1 \iff x = 0$$
**Effect algebra**

**Definition**

An *effect algebra* is a set $A$ together with the signature of partial functions

\[ A \times A \xrightarrow{\triangledown} A \leftarrow A \not\leftarrow I \]

where

- $(A, \triangledown, 0)$ is a commutative monoid,
- the following squares are pullbacks
Effect algebra

Definition

An effect algebra is a set $A$ together with the signature of partial functions

$$A \times A \overset{\triangledown}{\rightarrow} A \leftarrow A \overset{0}{\leftrightarrow} I$$

where

- $(A, \triangledown, 0)$ is a commutative monoid,
- the following strings are pulled
Effect algebra

Definition

An effect algebra is a set $A$ together with the signature of partial functions

$$A \times A \rightarrow A \leftarrow A \xleftarrow{\longrightarrow} I$$

where

- $(A, \sqcup, 0)$ is a commutative monoid,
- the following strings are pulled
string pullbacks!

:(
So what?

Why string diagrams of partial functions?

:(

(Mo) ef = (Fr) an
D. Pa and P.-M. Se

→ ↔ 1

spec ↔ sv
anti ↔ effect
modular ↔ Frob
Upshot
Task

Lift effect algebras from partial functions to dagger-compact categories
Extend string diagrams to general models. (Effect algebras are a simple test case.)
Outline

orthocomplement \iff one

special \iff single-valued

antispecial \iff effect algebra

modular \iff Frobenius

Upshot
Outline

orthocomplement $\leftrightarrow$ one

special $\leftrightarrow$ single-valued

antispecial $\leftrightarrow$ effect algebra

modular $\leftrightarrow$ Frobenius

Upshot
Context

- dagger-compact category $\mathcal{C}$

- classical monoid $A \otimes A \rightarrow A \leftarrow I$

- commutative monoid $A \otimes A \rightarrow A \leftarrow 0 \leftarrow I$
Orthocomplement operation

Definition

An orthocomplement with respect to \((A, \otimes, 0)\) is an operation \(\neg: A \rightarrow A\) such that for some \(\iota \in A\) and all \(x \in A\)

\[
\neg x \otimes x = \iota \quad \quad \quad \quad \neg \neg x = x
\]
Orthocomplement operation

**Definition**

An *orthocomplement* with respect to \((A, \ominus, 0)\) is an operation \(\neg : A \rightarrow A\) such that for some \(\iota \in A\)

\[
\begin{align*}
\neg \iota & = \neg \neg \iota \\
\iota & \ominus \neg \iota
\end{align*}
\]
Unbiased elements

Definition

An element $\iota \in A$ is *unbiased* with respect to $(A, \forall, 0)$ if

\[
\begin{array}{c}
\langle \\
A \\
\end{array}
\begin{array}{c}
\forall \\
\iota \\
\end{array}
\begin{array}{c}
\forall \\
\forall \\
A \\
\end{array} = \begin{array}{c}
\forall \\
\forall \\
A \\
\end{array}
\]
Orthocomplement

Proposition

For every commutative monoid $(A, \vee, 0)$ there is a bijection between

- orthocomplement operations $A \rightarrow A$ and
- unbiased vectors $I \rightarrow A$. 
Orthocomplement

Proof

The definition of an orthocomplement implies

\[ \neg \overline{A} = A \quad \text{and} \quad A \overline{A} = 0 \]
Orthocomplement

Proof

The definition of an orthocomplement implies

\[ A \rightarrow A \]

Then \( A \rightarrow A \) is an orthocomplement iff \( I \rightarrow A \) is unbiased.
Orthocomplemented monoid

Definition

An *orthocomplemented monoid* over a classical structure $A$ is a tuple $(A, \sqcup, 0, 1, \neg)$, where
- $(A, \sqcup, 0)$ is a commutative monoid,
- $I \overset{1}{\rightarrow} A$ is an unbiased vector, and
- $A \overset{\neg}{\rightarrow} A$ is the induced orthocomplementation.
De Morgan/Hadamard Laws

Proposition

\((A, \vee, 0, 1, \neg)\) is an orthocomplemented monoid iff 
\((A, \wedge, 1, 0, \neg)\) is an orthocomplemented monoid, where

\[
\text{De Morgan/Hadamard Laws}
\]

\[
\neg \iff 1
\]

\[
\text{spec} \iff \text{sv}
\]

\[
\text{anti} \iff \text{effect}
\]

\[
\text{modular} \iff \text{Frob}
\]

\[
\text{Upshot}
\]
Outline

orthocomplement ⟷ one

special ⟷ single-valued

antispecial ⟷ effect algebra

modular ⟷ Frobenius

Upshot
Convolution

Definition

Given

- a monoid \((A, \mu, \iota)\)
- a comonoid \((A, \lambda, \epsilon)\)

the induced

- convolution monoid \((\mathbb{C}(A, A), \star, \iota \circ \epsilon)\)

is defined by

\[
\begin{array}{c}
A \\
\downarrow f \\
A \\
\star \\
A \\
\downarrow g \\
A \\
\end{array}
\quad \star 
\quad \begin{array}{c}
A \\
\downarrow f \\
A \\
\downarrow \lambda \\
A \\
\downarrow g \\
A \\
\end{array}
\]

\[
\begin{array}{c}
A \\
\downarrow \mu \\
A \\
\end{array}
\quad = 
\quad \begin{array}{c}
A \\
\downarrow \mu \\
A \\
\end{array}
\]
Specialties

Definition

A convolution algebra \((A, \mu, \iota, \lambda, \epsilon)\) is called

i. *special* if \(\text{id} \star \text{id}\) is unitary, and

ii. *antispecial* if \(\text{id} \star \text{id}\) is a scaled projector.
Explanation

Recall that $e \in \mathbb{C}(A, A)$ is a

i. \textit{unitary} when $e \circ e^\dagger = e^\dagger \circ e = \text{id}$;

ii. \textit{scaled projector} when $e = a \circ b^\dagger$, $a, b \in \mathbb{C}(A)$. 
Specialties

Cayley

A vector $b \in \mathbb{C}(B)$ is

i. _unbiased_ when $\gamma b$ is unitary;

ii. a _basis_ vector when $\gamma b$ is pure projector,

where

is the Cayley representation
Convolution preorder

Definition

\[ f \leq g \iff \exists \ell \in \mathcal{C}(A, B). \ f \star \ell = g \]
Definition

A morphism $f \in \mathcal{C}(A, B)$ in a dagger-compact category $\mathcal{C}$ is said to be

i. **total** if

$$id_A \leq f^\dagger \circ f$$

ii. **single-valued** (or a partial map) if

$$f \circ f^\dagger \leq id_B$$

iii. a **map** if it is total and single-valued.
Maps

Proposition

Let $\mathcal{C}$ be dagger-compact category with chosen classical structures. The induced convolution preorders make it into a *cartesian bicategory* (à la Carboni-Walters).

Then for every $f \in \mathcal{C}(A, B)$ holds

i. $f$ is total if and only if

$$!_B \circ f = !_A$$

ii. $f$ is partial map if and only if

$$\Delta_B \circ f = (f \otimes f) \circ \Delta_A$$

iii. $f$ is a map iff it is a comonoid homomorphism.
Special $\iff$ single-valued

Proposition

$(A, \bigodot, 0)$ single-valued $\iff (A, \bigodot, 0, \bigodot^\dagger, 0^\dagger)$ special
Outline

orthocomplement ↔ one

special ↔ single-valued

antispecial ↔ effect algebra

modular ↔ Frobenius

Upshot
General effect algebra

Definition

Let $\mathcal{C}$ be dagger-compact category with classical structures$^1$. A general effect algebra is

- $A \times A \xrightarrow{\otimes} A \leftarrow A \xleftarrow{\ll} I$ — single-valued diagram
- $(A, \otimes, 0)$ — commutative monoid,
- such that

$^1$thus a cartesian bicategory
Superspecial

Definition

An orthocomplemented algebra \((A, \bigvee, \bigwedge, 0, 1, \neg)\) is said to be *superspecial* if

(a) the convolution algebra \((A, \bigvee, 0, \bigvee^\perp, 0^\perp)\) is special
(b) the convolution algebra \((A, \bigvee, 0, \bigvee^\perp, 1^\perp)\) is antispecial.
Proposition

\((A, \ominus, \oslash, 0, 1, \neg)\) is superspecial iff 
\((A, \ominus, 0, 1, \neg)\) is a general effect algebra.
Superspecial $\leftrightarrow$ effect algebra

Proof idea

$x \oplus y = 1 \iff x = \neg y \quad \land \quad x \oplus 1 = 1 \iff x = 0$

$x \oplus y = u \land x \ominus y = v \iff u = 1 \land v = 0$
Superspecial $\leftrightarrow$ effect algebra

Proof (1)

Since

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\oplus} & A \\
\downarrow & & \downarrow \\
A \otimes A & \xrightarrow{\ominus} & A \\
\end{array}
\]

is a pullback
Superspecial $\leftrightarrow$ effect algebra

Proof (1)

...it follows that

$A \xrightarrow{!} I \iff A \xrightarrow{!} I \iff A \otimes A \xrightarrow{!} I$

\[
\begin{array}{ccc}
A & \xrightarrow{!} & I \\
\downarrow & & \downarrow \\
A \otimes A & \xrightarrow{\otimes} & A \\
\langle \text{id}, \neg \rangle & & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{!} & I \\
\downarrow & & \downarrow \\
A \otimes A & \xrightarrow{\otimes} & A \\
\langle \text{id}, \neg \rangle & & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\otimes} & A \\
\langle \text{id}, \neg \rangle & & \langle \pi_0, \neg, \pi_1, \neg \rangle \\
\end{array}
\]

\[
\begin{array}{ccc}
A \otimes A \otimes A \otimes A & \xrightarrow{\otimes \otimes \otimes} & A \otimes A \\
\langle 1, 0 \rangle & & \langle 1, 0 \rangle \\
\end{array}
\]
Superspecial $\leftrightarrow$ effect algebra

Proof (2)

where

\[ \omega = \]
Superspecial $\iff$ effect algebra

Proof (3)

The uniqueness part of the pullback condition is

\[
\begin{array}{ccc}
\exists
\end{array}
\]

— which transforms to the antispecial condition.

\[
\begin{array}{ccc}
1 & \ 0 & \omega
\end{array}
\]
Superspecial \(\leftrightarrow\) effect algebra

Proof of the left-hand equation of (1)

By associativity + single-valuedness, the LHS becomes
Superspecial $\leftrightarrow$ effect algebra

Proof of the left-hand equation of (1)

By associativity + single-valuedness, the LHS becomes

The RHS is the path around the pullback.
Superspecial $\leftrightarrow$ effect algebra

Proof of the left-hand equation of (1)

Moving the $\neg$s to reduce $\bigvee$ to $\bigvee$
Superspecial $\leftrightarrow$ effect algebra

Proof of the left-hand equation of (1)

Moving the $\neg$s to reduce $\wedge$ to $\vee$

The result follows using the other two pullbacks.
Outline

orthocomplement $\iff$ one

special $\iff$ single-valued

antispecial $\iff$ effect algebra

modular $\iff$ Frobenius

Upshot
Modularity in lattices

\[ x \leq z \quad \Rightarrow \quad (x \lor y) \land z = x \lor (y \land z) \]
Modularity in effect algebras in Pfn

\[ x \leq -y \leq z \implies (x \circ y) \circ z = x \circ (y \circ z) \]
Question

How do you write conditional equations in string diagrams?
Question

How do you write conditional equations in string diagrams?

Answer

For partial maps, you can use convolutions!
Modularity in effect algebras in $\mathbb{C}$
Lemma
For partial maps $f, g \in C_s(A, B)$

\[
\begin{align*}
\begin{array}{cc}
\begin{array}{c}
\begin{tikzpicture}
\node (f) at (0,0) {$f$};
\node (g) at (1,0) {$g$};
\node (gf) at (1,1) {$g^*$};
\node (gf2) at (0,1) {$f$};
\end{tikzpicture}
\end{array} & = \begin{array}{c}
\begin{tikzpicture}
\node (g) at (1,0) {$g$};
\node (gf) at (1,1) {$g^*$};
\node (gf2) at (0,1) {$f$};
\end{tikzpicture}
\end{array} \end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{cc}
\begin{array}{c}
\begin{tikzpicture}
\node (f) at (0,0) {$f$};
\node (g) at (1,0) {$g$};
\node (fg) at (1,1) {$f$};
\end{tikzpicture}
\end{array} & = \begin{array}{c}
\begin{tikzpicture}
\node (f) at (0,0) {$f$};
\node (g) at (1,0) {$g$};
\node (fg) at (1,1) {$f$};
\end{tikzpicture}
\end{array} \end{array}
\end{align*}
\]

\[\iff f \leq g\]
Frobenius condition
Equivalent forms of the Frobenius condition

\[
(Mo) \text{ ef } = (Fr) \text{ an }\]

D. Pa and P.-M. Se

\[
\rightarrow \leftrightarrow 1
\]

\[
\text{spec} \leftrightarrow \text{sv}
\]

\[
\text{anti} \leftrightarrow \text{effect}
\]

modular \leftrightarrow \text{Frob}

Upshot
Modularity = Frobenius

Proposition

A superspecial algebra \((A, \odot, \ominus, 0, 1, \neg)\) over a self-dual object \(A\) in a dagger-compact category \(\mathcal{C}\) satisfies the Frobenius condition if and only if it is modular.
Modularity = Frobenius

Proof
Outline

orthocomplement $\leftrightarrow$ one

special $\leftrightarrow$ single-valued

antispecial $\leftrightarrow$ effect algebra

modular $\leftrightarrow$ Frobenius

Upshot
Moral: Strings utilitarianism
Moral: Strings utilitarianism

- Picturalism: Ode to doubling
Moral: Strings utilitarianism

- **Picturalism**: Ode to doubling

- **Geometry of abstraction**: Unknot the strings in algorithms
Moral: Strings utilitarianism

- **Picturalism:** Ode to doubling

- **Geometry of abstraction:** Unknot the strings in algorithms

- **von Frobenius:** Pull the strings in boxes in boxes