(Modular) effect algebras are equivalent to (Frobenius) antispecial algebras

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Effect algebra

Definition

An *effect algebra* is a set A together with the signature of partial functions

$$A \times A \xrightarrow{\oslash} A \xleftarrow{\neg} A \underset{1}{\overset{0}{\leftleftarrows}} I$$

where

- $(A, \oslash, 0)$ is a commutative monoid,
- the following conditions are satisified for all $x, y \in A$

$$x \oslash y = 1 \iff x = \neg y$$
$$x \oslash 1 = 1 \iff x = 0$$

Effect algebra

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where

- $(A, \ovee, 0)$ is a commutative monoid,
- the following squares are pullbacks

$$\begin{array}{ccc} A & \xrightarrow{!} & I \\ {\scriptstyle \langle \mathrm{id}, \neg \rangle}\downarrow & \lrcorner & \downarrow{\scriptstyle 1} \\ A \otimes A & \xrightarrow[\ovee]{} & A \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{\mathrm{id}} & I \\ {\scriptstyle \langle 0,0 \rangle}\downarrow & \lrcorner & \downarrow{\scriptstyle 0} \\ A \otimes A & \xrightarrow[\ovee]{} & A \end{array}$$

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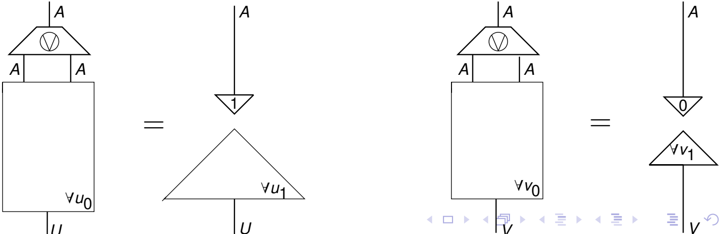

Effect algebra

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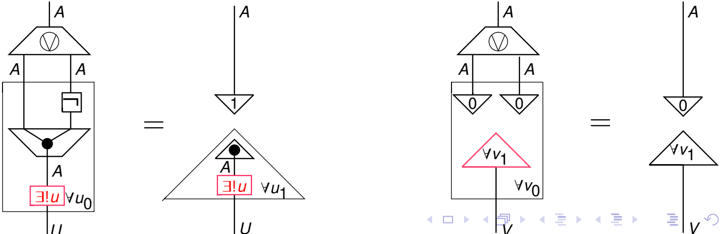

string pullbacks!

:)

So what?

Why string diagrams of partial functions?

:(

Task

Lift effect algebras

from partial functions

to dagger-compact categories

Upshot

Extend string diagrams to general models.

(Effect algebras are a simple test case.)

Outline

orthocomplement ⇝ one

special ⇝ single-valued

antispecial ⇝ effect algebra

modular ⇝ Frobenius

Upshot

Outline

orthocomplement ⇝ one

special ⇝ single-valued

antispecial ⇝ effect algebra

modular ⇝ Frobenius

Upshot

Context

- dagger-compact category $\mathbb{C}$
- classical monoid $A \otimes A \xrightarrow{\blacktriangledown} A \xleftarrow{!} I$
- commutative monoid $A \otimes A \xrightarrow{\ovee} A \xleftarrow{0} I$

Orthocomplement operation

Definition

An *orthocomplement* with respect to $(A, \varovee, 0)$ is an operation $\neg : A \longrightarrow A$ such that for some $\iota \in A$ and all $x \in A$

$$\neg x \varovee x = \iota \qquad \neg\neg x = x$$

Orthocomplement operation

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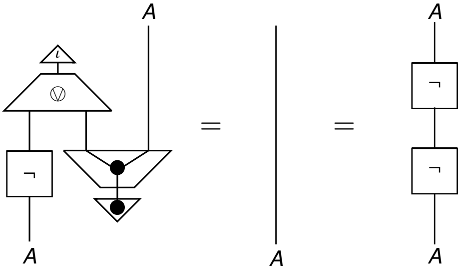
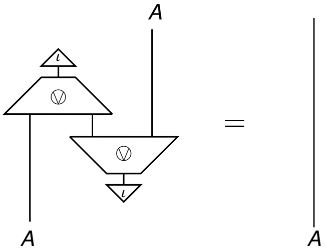


Unbiased elements

Definition

An element $\iota \in A$ is *unbiased* with respect to $(A, \bigtriangledown, 0)$ if



Orthocomplement

Proposition

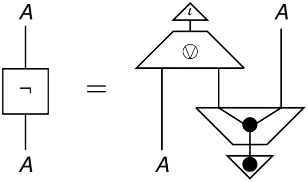
For every commutative monoid $(A, \oslash, 0)$ there is a bijection between

- orthocomplement operations $A \xrightarrow{\neg} A$ and
- unbiased vectors $I \xrightarrow{\iota} A$.

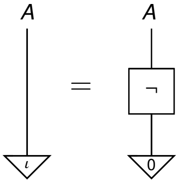
Orthocomplement

Proof

The definition of an orthocomplement implies



and



Orthocomplement

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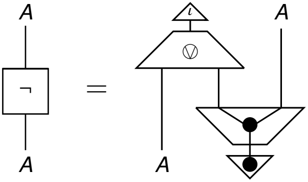

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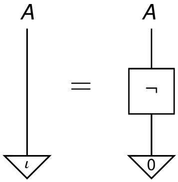


Then $A \xrightarrow{\neg} A$ is an orthocomplement iff $I \xrightarrow{\iota} A$ is unbiased.

Orthocomplemented monoid

Definition

An *orthocomplemented monoid* over a classical structure A is a tuple $(A, \owedge, 0, 1, \neg)$, where

- $(A, \owedge, 0)$ is a commutative monoid,
- $I \xrightarrow{1} A$ is an unbiased vector, and
- $A \xrightarrow{\neg} A$ is the induced orthocomplementation.

De Morgan/Hadamard Laws

Proposition

$(A, \ovee, 0, 1, \neg)$ is an orthocomplemented monoid iff
$(A, \owedge, 1, 0, \neg)$ is an orthocomplemented monoid, where

Outline

Convolution

Definition

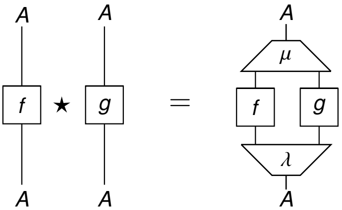
Given

- a monoid (A, μ, ι)
- a comonoid (A, λ, ϵ)

the induced

- convolution monoid $\left(\mathbb{C}(A, A), \star, \iota \circ \epsilon\right)$

is defined by



Specialties

Definition

A convolution algebra $(A, \mu, \iota, \lambda, \epsilon)$ is called

i. *special* if $\mathrm{id}\star\mathrm{id}$ is unitary, and
ii. *antispecial* if $\mathrm{id}\star\mathrm{id}$ is a scaled projector.

Specialties

Explanation

Recall that $e \in \mathbb{C}(A, A)$ is a

i. *unitary* when $e \circ e^{\ddagger} = e^{\ddagger} \circ e = \mathrm{id}$;

ii. *scaled projector* when $e = a \circ b^{\ddagger}$, $a, b \in \mathbb{C}(A)$.

Specialties

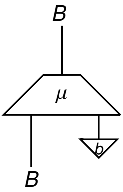
Cayley

A vector $b \in \mathbb{C}(B)$ is

i. *unbiased* when Υb is unitary;

ii. a *basis* vector when Υb is pure projector,

where

$\Upsilon b \quad =$ 

is the Cayley representation

Convolution preorder

Definition

$$f \leq g \iff \exists \ell \in \mathbb{C}(A, B).\ f \star \ell = g$$

Maps

Definition

A morphism $f \in \mathbb{C}(A, B)$ in a dagger-compact category $\mathbb{C}$ is said to be

i. *total* if

$$\mathrm{id_A} \quad \leq \quad f^\ddagger \circ f$$

ii. *single-valued* (or a *partial map*) if

$$f \circ f^\ddagger \quad \leq \quad \mathrm{id_B}$$

iii. a *map* if it is total and single-valued.

Maps

Proposition

Let $\mathbb{C}$ be dagger-compact category with chosen classical structures. The induced convolution preorders make it into a *cartesian bicategory* (à la Carboni-Walters).

Then for every $f \in \mathbb{C}(A, B)$ holds

i. f is total if and only if

$$!_B \circ f \quad = \quad !_A$$

ii. f is partial map if and only if

$$\blacktriangle_B \circ f \quad = \quad (f \otimes f) \circ \blacktriangle_A$$

iii. f is a map iff it is a comonoid homomorphism.

Special $\leftrightsquigarrow$ single-valued

Proposition

$(A, \bigtriangledown, 0)$ single-valued $\iff$ $(A, \bigtriangledown, 0, \bigtriangledown^{\ddagger}, 0^{\ddagger})$ special

Outline

orthocomplement ⇝ one

special ⇝ single-valued

antispecial ⇝ effect algebra

modular ⇝ Frobenius

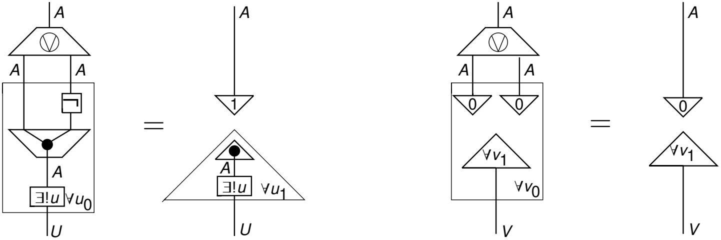
Upshot

General effect algebra

Definition

Let $\mathbb{C}$ be dagger-compact category with classical structures[1]. A *general effect algebra* is

- $A \times A \xrightarrow{\varovee} A \xleftarrow{\neg} A \underset{1}{\overset{0}{\leftleftarrows}} I$ — single-valued diagram
- $(A, \varovee, 0)$ — commutative monoid,
- such that



[1]thus a cartesian bicategory

Superspecial

Definition

An orthocomplemented algebra $(A, \ovee, \owedge, 0, 1, \neg)$ is said to be *superspecial* if

(a) the convolution algebra $(A, \ovee, 0, \ovee^{\ddagger}, 0^{\ddagger})$ is special (the convolution algebra $(A, \owedge, 1, \owedge^{\ddagger}, 1^{\ddagger})$ is special), and

(b) the convolution algebra $(A, \ovee, 0, \owedge^{\ddagger}, 1^{\ddagger})$ is antispecial.

Superspecial $\leftrightsquigarrow$ effect algebra

Proposition

$(A, \varovee, \varowedge, 0, 1, \neg)$ is superspecial iff
$(A, \varovee, 0, 1, \neg)$ is a general effect algebra.

Superspecial ⇝ effect algebra

Proof idea

$$x \varovee y = 1 \iff x = \neg y \qquad \wedge \qquad x \varovee 1 = 1 \iff x = 0$$

$$\Updownarrow$$

$$x \varovee y = u \ \wedge \ x \varowedge y = v \iff u = 1 \ \wedge \ v = 0$$

Superspecial ⇝ effect algebra

Proof (1)

Since

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\ovee} & A \\ {\scriptstyle \neg\otimes\neg}\downarrow & \lrcorner & \downarrow{\scriptstyle \neg} \\ A\otimes A & \xrightarrow[\owedge]{} & A \end{array}$$

is a pullback

Superspecial ⇝ effect algebra

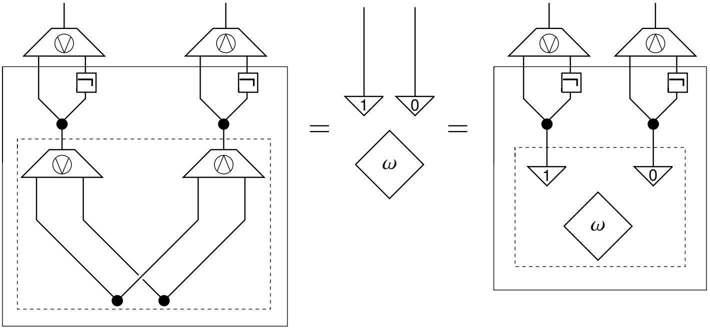
Proof (1)

. . . it follows that

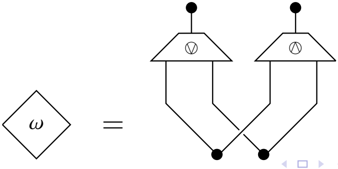
$$
\begin{array}{ccc}
A & \xrightarrow{!} & I \\
{\scriptstyle\langle \mathrm{id},\neg\rangle}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 1} \\
A\otimes A & \xrightarrow[\ovee]{} & A
\end{array}
\quad\Longleftrightarrow\quad
\begin{array}{ccc}
A & \xrightarrow{!} & I \\
{\scriptstyle\langle \mathrm{id},\neg\rangle}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle 0} \\
A\otimes A & \xrightarrow[\owedge]{} & A
\end{array}
\quad\Longleftrightarrow\quad
\begin{array}{ccc}
A\otimes A & \xrightarrow{!} & I \\
{\scriptstyle\langle \pi_0,\neg,\pi_1,\neg\rangle}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle\langle 1,0\rangle} \\
A\otimes A\otimes A\otimes A & \xrightarrow[\ovee\otimes\owedge]{} & A\otimes A
\end{array}
$$

Superspecial ⇝ effect algebra

Proof (2)



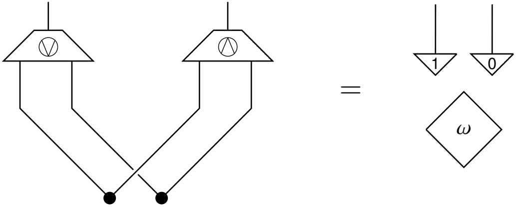
where



Superspecial ⇝ effect algebra

Proof (3)

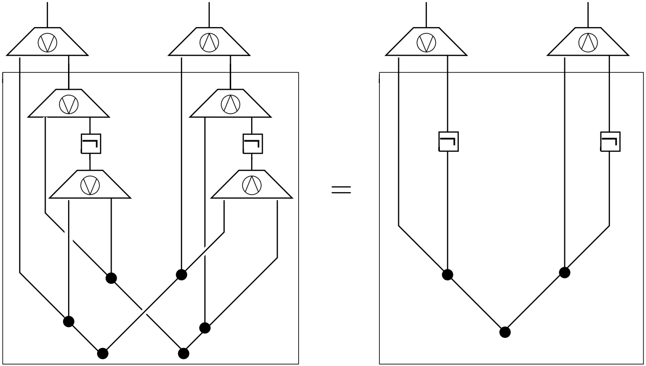
The uniqueness part of the pulllback condition is



— which transforms to the antispecial condition.

Proof of the left-hand equation of (1)

By associativity + single-valuedness, the LHS becomes



Superspecial ⇝ effect algebra

Proof of the left-hand equation of (1)

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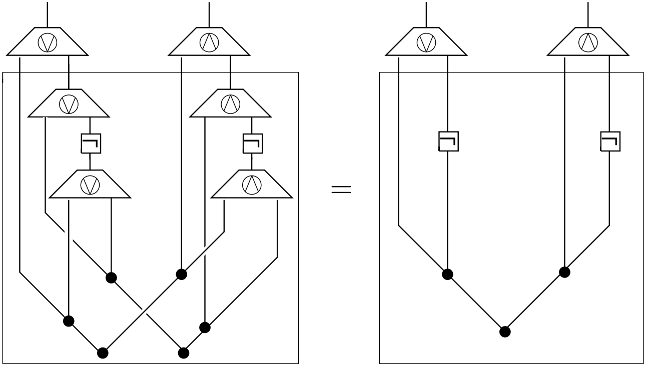
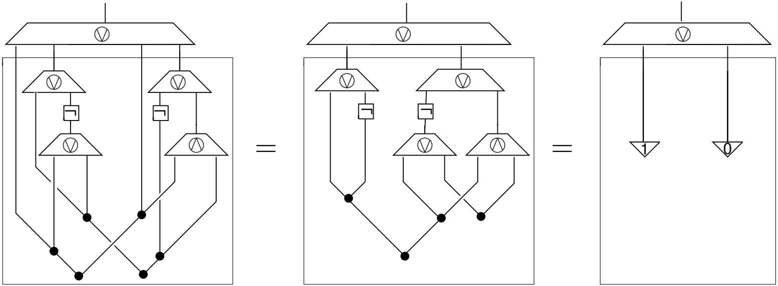


The RHS is the path around the pullback.

Superspecial ⇝ effect algebra

Proof of the left-hand equation of (1)

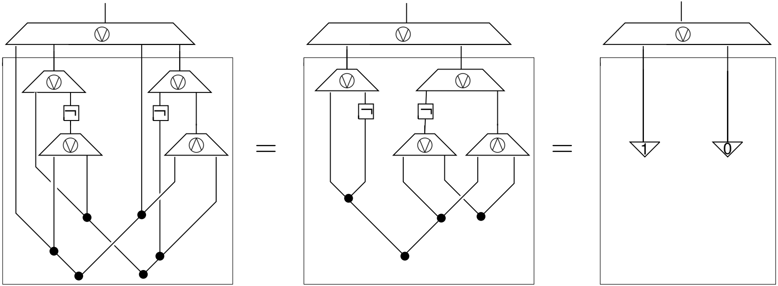
Moving the ¬s to reduce $\bigcirc\!\!\!\!\wedge$ to $\bigcirc\!\!\!\!\vee$



Superspecial ⇝ effect algebra

Proof of the left-hand equation of (1)

Moving the ¬s to reduce ∧⃝ to ∨⃝



The result follows using the other two pullbacks.

Outline

Modularity in lattices

$$x \leq z \implies (x \vee y) \wedge z = x \vee (y \wedge z)$$

Modularity in effect algebras in Pfn

$$x \leq \neg y \leq z \implies (x \ovee y) \owedge z = x \ovee (y \owedge z)$$

Question

How do you write conditional equations in string diagrams?

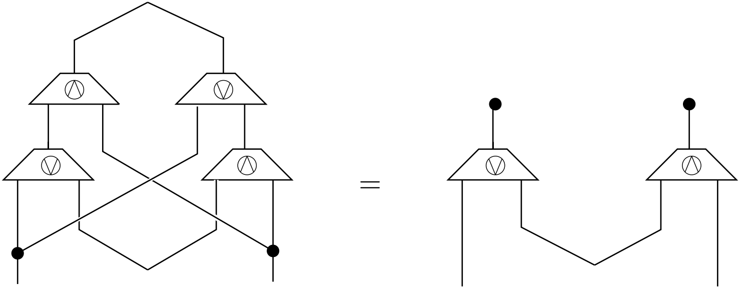
Question

How do you write conditional equations in string diagrams?

Answer

For partial maps, you can use convolutions!

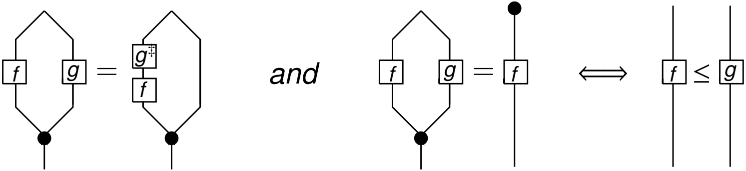
Modularity in effect algebras in $\mathbb{C}$



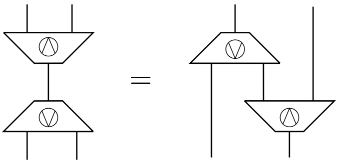
. . . using

Lemma

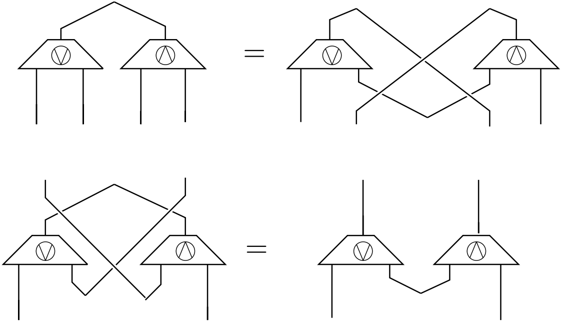
For partial maps $f, g \in \mathbb{C}_s(A, B)$



Frobenius condition



Equivalent forms of the Frobenius condition



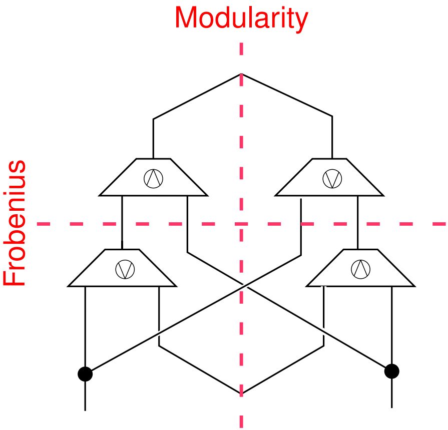
Modularity = Frobenius

Proposition

A superspecial algebra $(A, \varovee, \varowedge, 0, 1, \neg)$ over a self-dual object A in a dagger-compact category $\mathbb{C}$ satisfies the Frobenius condition if and only if it is modular.

Modularity = Frobenius

Proof



Outline

orthocomplement ⇜⇝ one

special ⇜⇝ single-valued

antispecial ⇜⇝ effect algebra

modular ⇜⇝ Frobenius

Upshot

Moral: Strings utilitarianism

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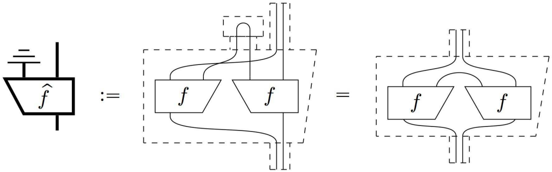
- **Picturalism:** Ode to doubling



Moral: Strings utilitarianism

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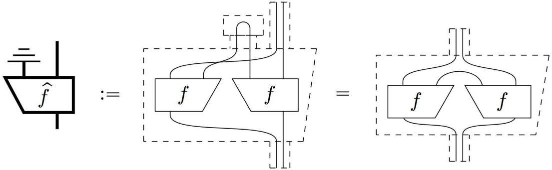
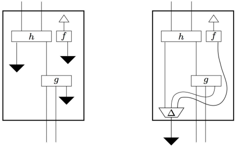


- **Geometry of abstraction:** Unknot the strings in algorithms



Moral: Strings utilitarianism

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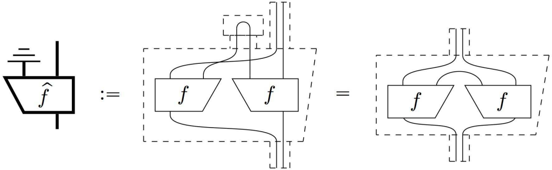

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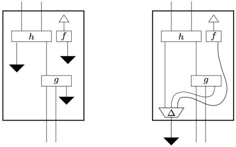

- **von Frobenius:** Pull the strings in boxes in boxes