A Simplified Stabilizer ZX-calculus

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Outline

Background

Simplifying the ZX-calculus

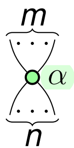
Which rewrite rules are necessary

Conclusions

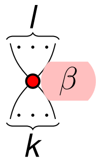
Outline

Elements of ZX-calculus diagrams

- green nodes with *n* inputs and *m* outputs, $\alpha \in (-\pi, \pi]$



- red nodes with *n* inputs and *m* outputs, $\beta \in (-\pi, \pi]$



- Hadamard nodes with one input and one output

- star nodes with no inputs or outputs

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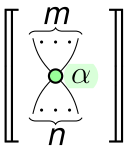

$$:= |0\rangle^{\otimes m} \langle 0|^{\otimes n} + e^{i\alpha} |1\rangle^{\otimes m} \langle 1|^{\otimes n},$$

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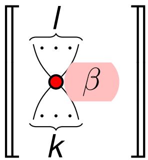


$$:= |+\rangle^{\otimes m} \langle +|^{\otimes n} + e^{i\beta} |-\rangle^{\otimes m} \langle -|^{\otimes n},$$

- Hadamard nodes with one input and one output

$$[\![\text{H}]\!] := |+\rangle \langle 0| + |-\rangle \langle 1|$$

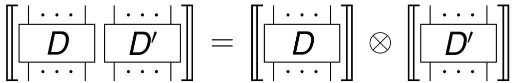
- star nodes with no inputs or outputs

$$[\![\bigstar]\!] := \frac{1}{2}$$

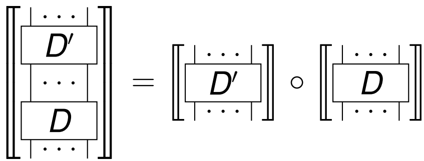
Composite diagrams

For arbitrary diagrams *D* and *D*′:

- parallel composition corresponds to tensor product:



- sequential composition corresponds to matrix product:



(where the number of outputs of *D* must be equal to the number of inputs of *D*′)

Stabilizer quantum mechanics

Consists of:

- preparation of qubits in state $|0\rangle$
- Clifford unitaries, generated by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \; H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; C_X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- measurements in computational basis

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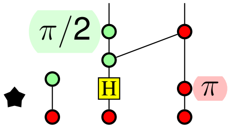
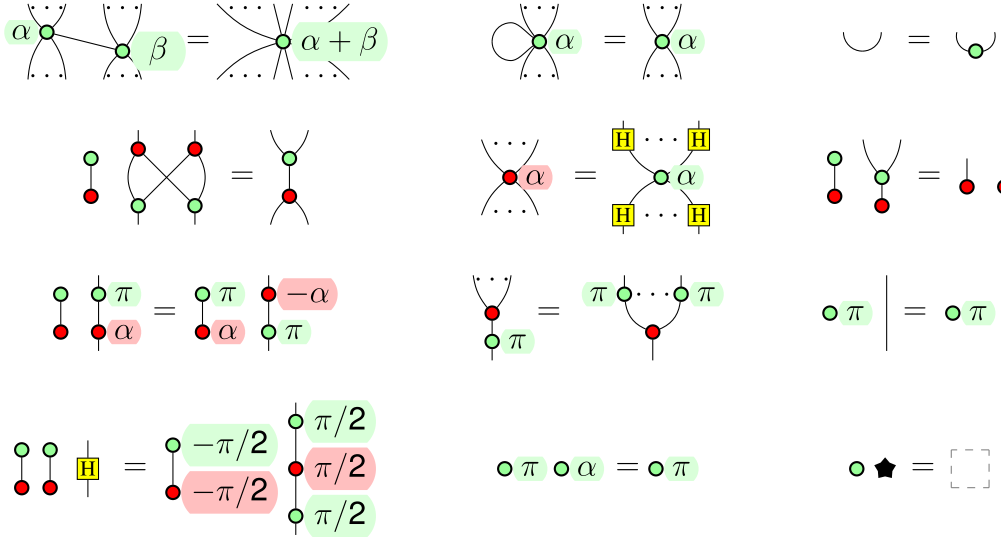
In ZX-calculus:

- diagrams in which all phase angles are integer multiples of $\pi/2$
- e.g.



Rules of the stabilizer ZX-calculus



Meta rules:

- Only the topology matters.
- All the rules above also hold upside-down and/or with colours swapped.

Completeness and minimality

Definition

A graphical language for QM is *complete* if any two diagrams representing the same matrix are equal according to the graphical rules, i.e.:

$$[\![D_1]\!] = [\![D_2]\!] \quad \Longrightarrow \quad D_1 = D_2$$

Theorem (B, 2012/2015)

The stabilizer ZX*-calculus is complete.*

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Definition

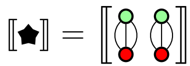
A set of rules for a graphical language is *minimal* if no rule can be derived from the others.

Can we find a minimal complete rule set for the ZX*-calculus?*

Outline

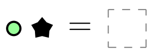
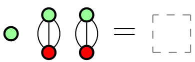
Simplifying the notation for scalars

We have:

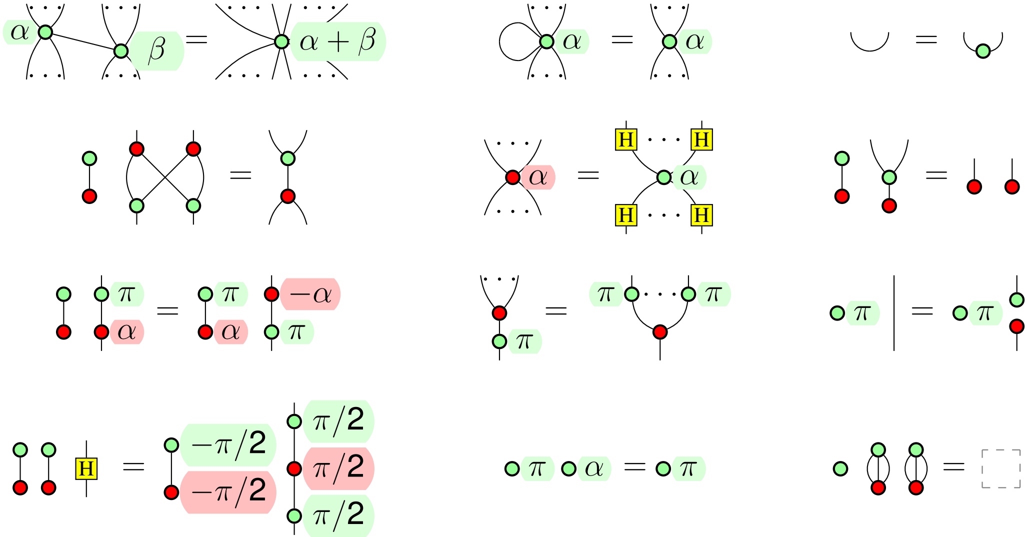


so the star node ★ is not necessary.

Replace occurence in rewrite rules:

 becomes 

Removing derivable rules (and modifying others)



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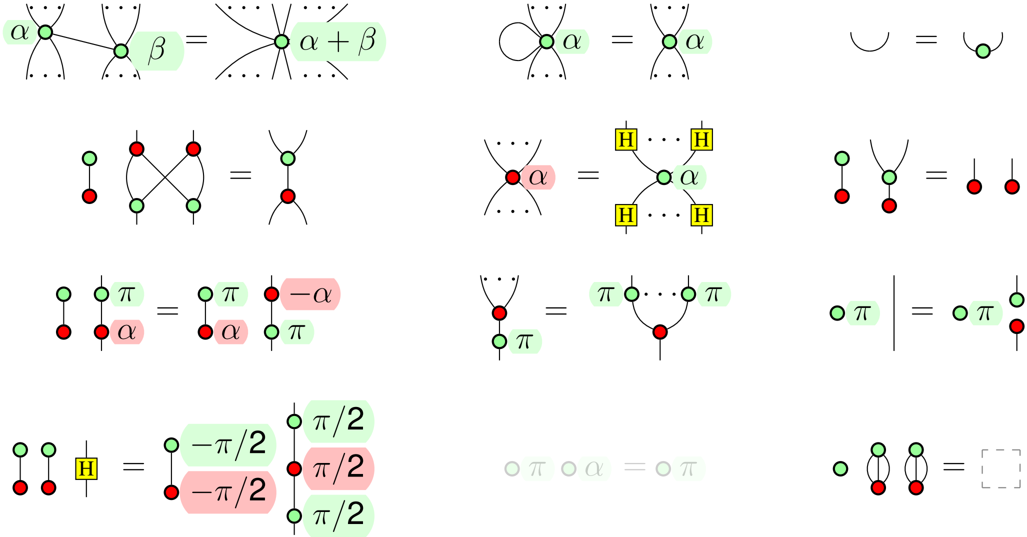

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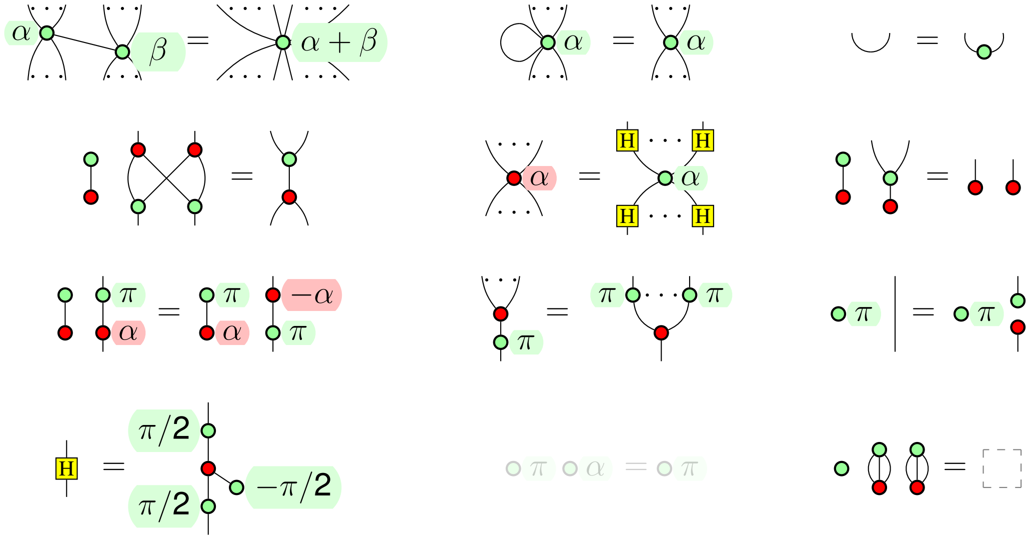

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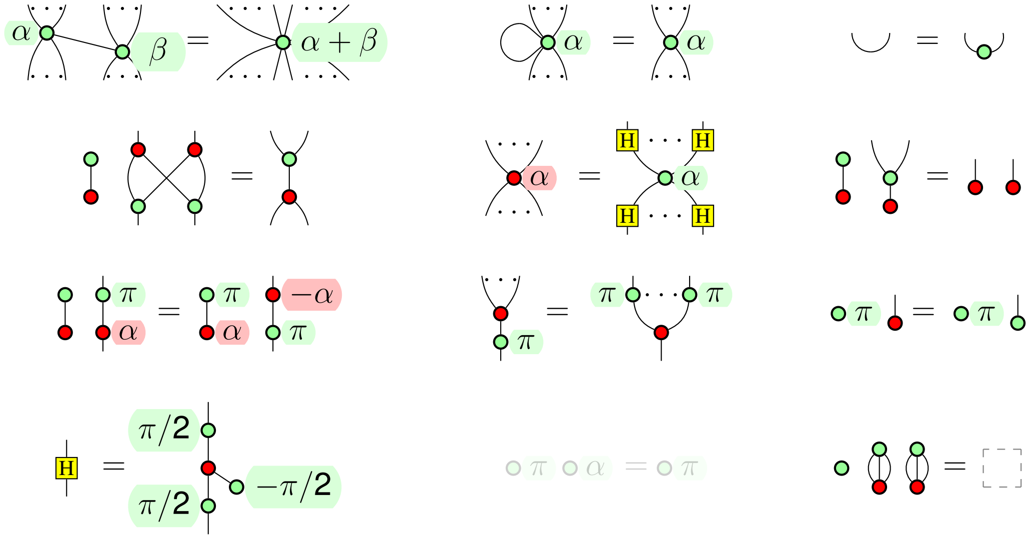

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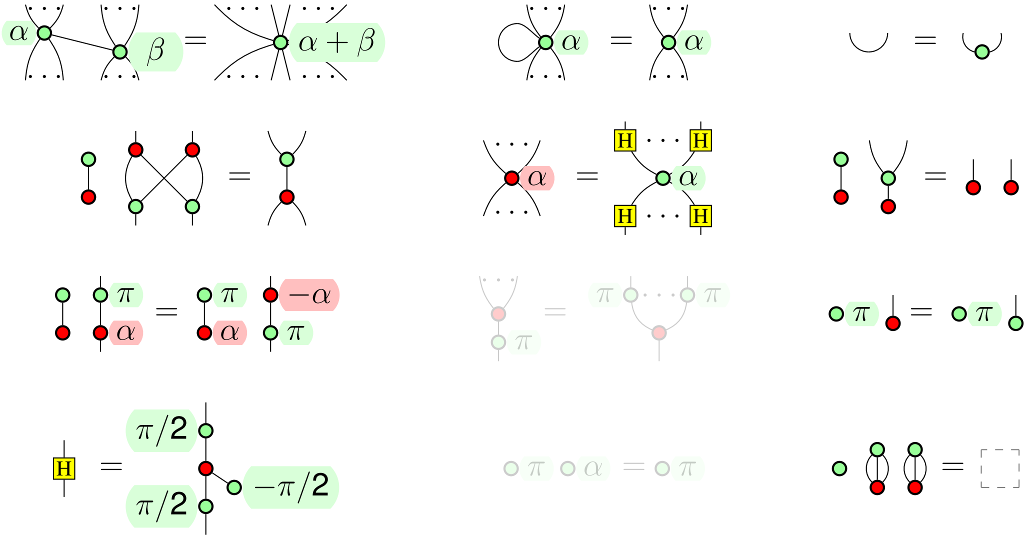

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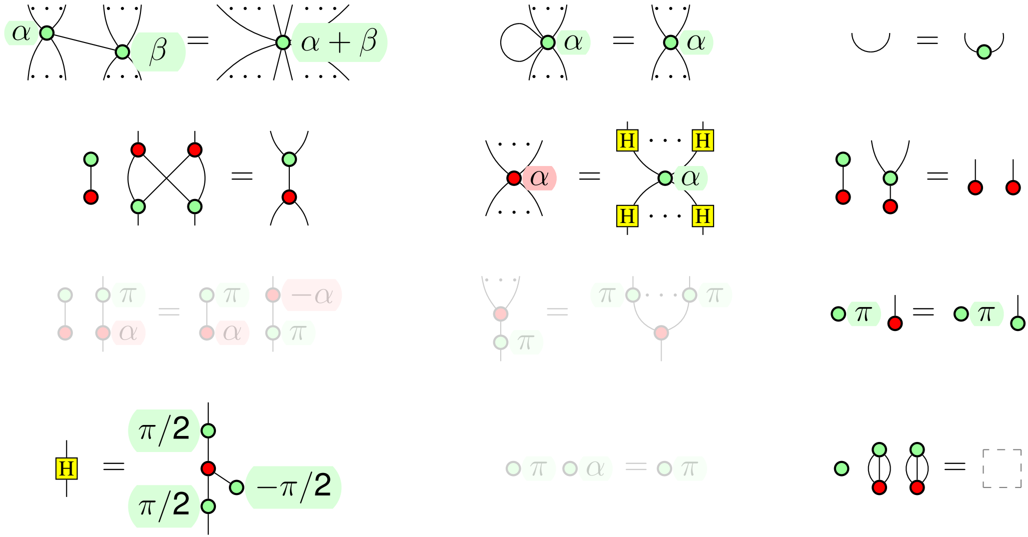

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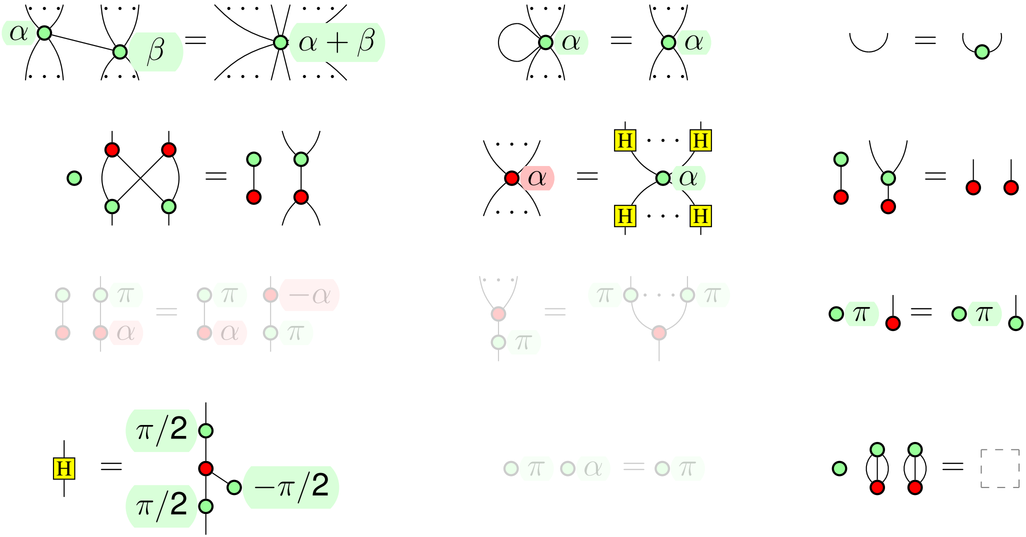

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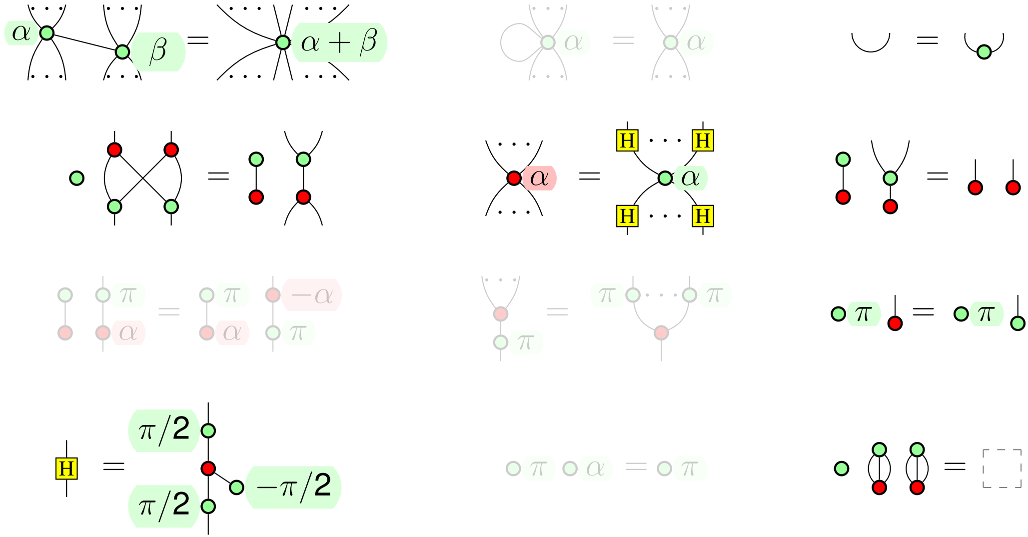

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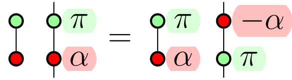
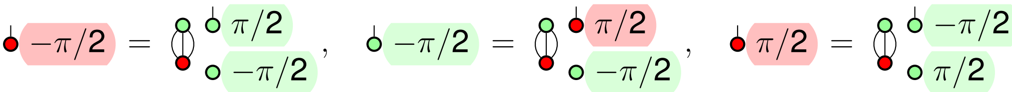
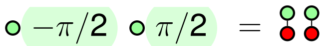
Example: deriving the π-commutation rule

To show:  for $\alpha \in \{0, \pm\pi/2, \pi\}$

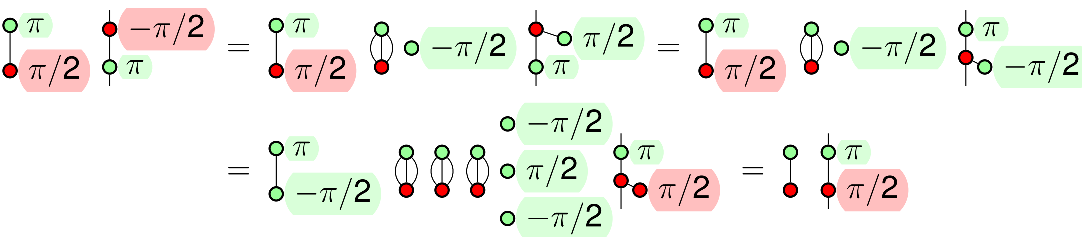
- Derive equalities about states with phases $\pm\pi/2$:



- Use these to show:

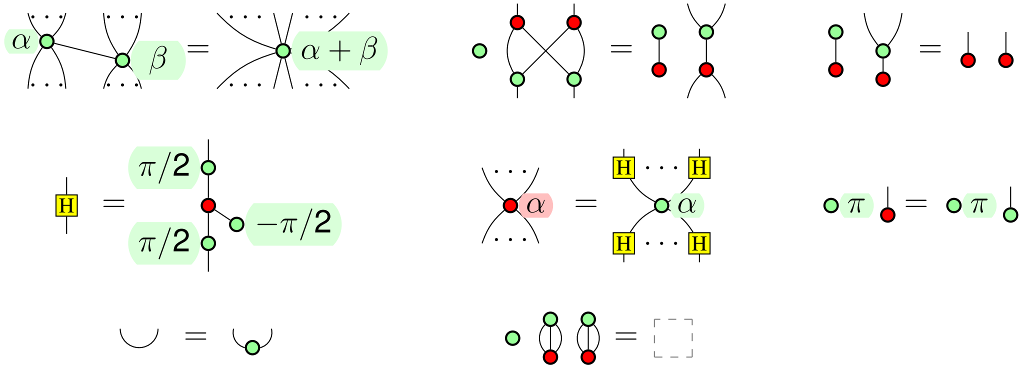


- Prove the desired equality for each value of α in turn (here: $\alpha = \pi/2$):



This derivation only works within stabilizer QM.

Removing colour-swapped and upside-down duplicates



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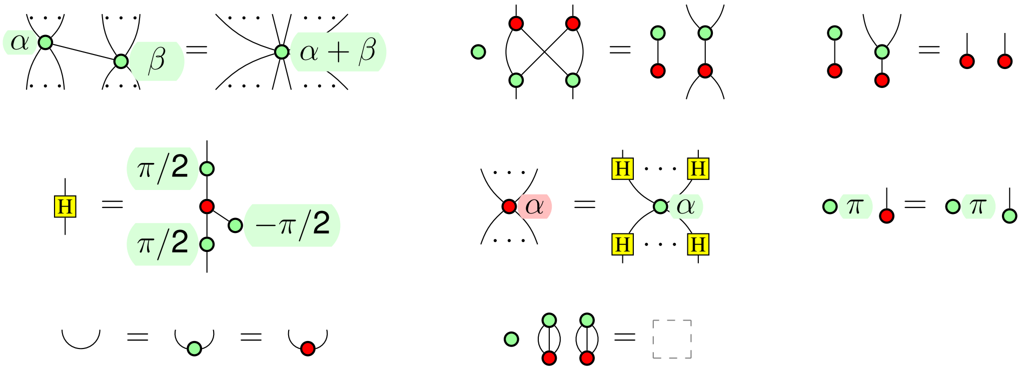

Meta rules:

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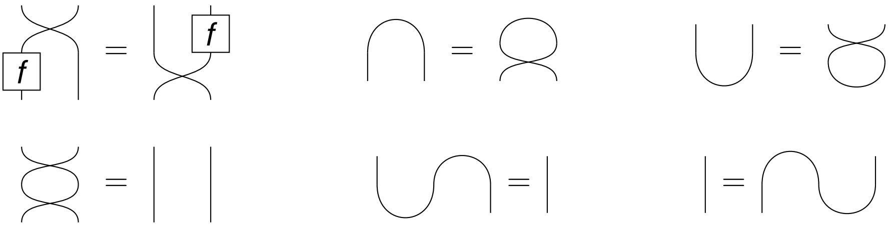
The topology meta rule

Combines two different sets of properties:

- axioms of a symmetric compact closed category, i.e. existence of wire crossing, cup, and cap, satisfying:



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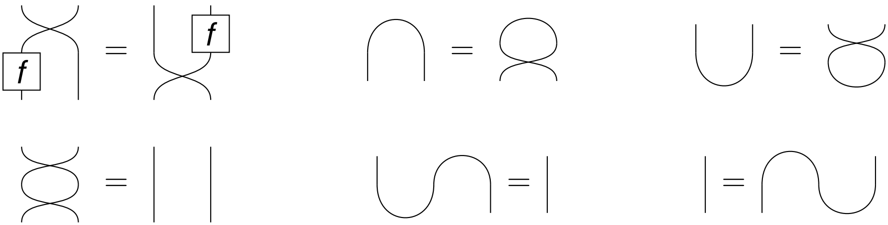
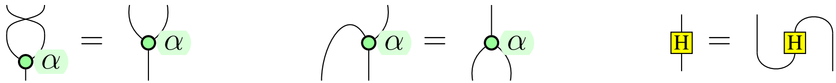
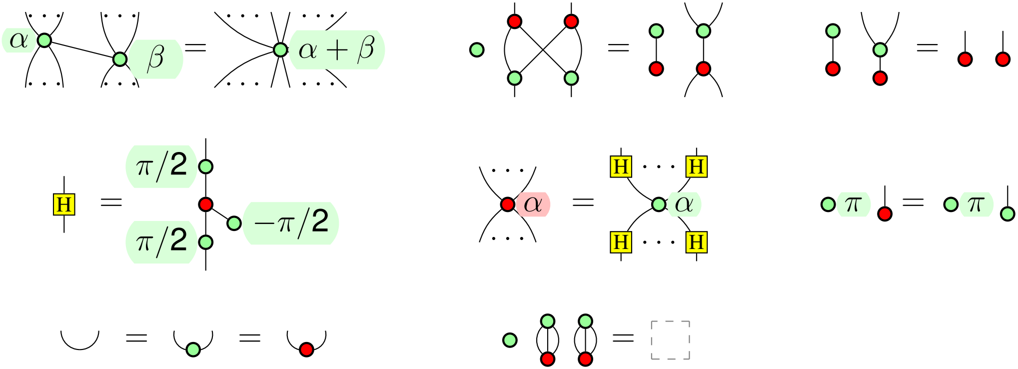


- basic diagram components are invariant under interchange of two inputs or outputs, as well as under bending inputs into outputs or conversely, e.g.:



Simplifying the topology meta rule



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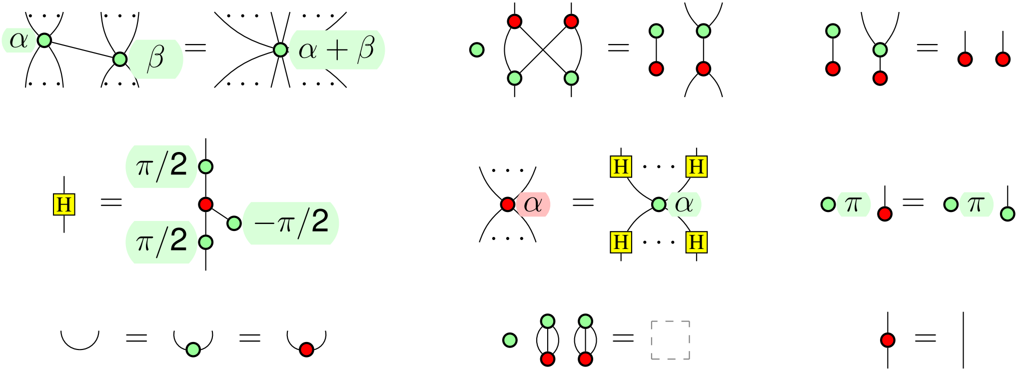
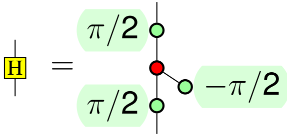


Axioms of a symmetric compact closed category

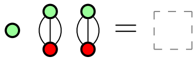
Outline

- Background
- Simplifying the ZX-calculus
- Which rewrite rules are necessary
- Conclusions

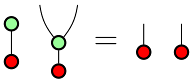
Some necessary rules



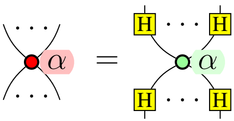
proof from [Duncan & Perdrix, 2009/2014] carries over with slight modifications



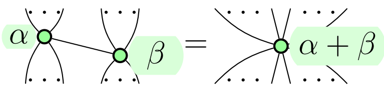
only rule to match the empty diagram



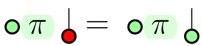
only rule to map connected outputs to disconnected ones



only rule to match red nodes of degree ≥ 4



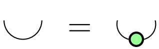
only rule that can transform nodes of degree $\geq$ 4 into diagrams containing only lower-degree nodes



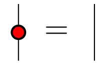
proof uses an alternative interpretation functor that ‘doubles up’ spiders in different colours (see arXiv:1602.04744)

Wires and nodes

The only rules to map between a diagram containing nodes and a diagram containing only wires are the cup rule and the identity rule:

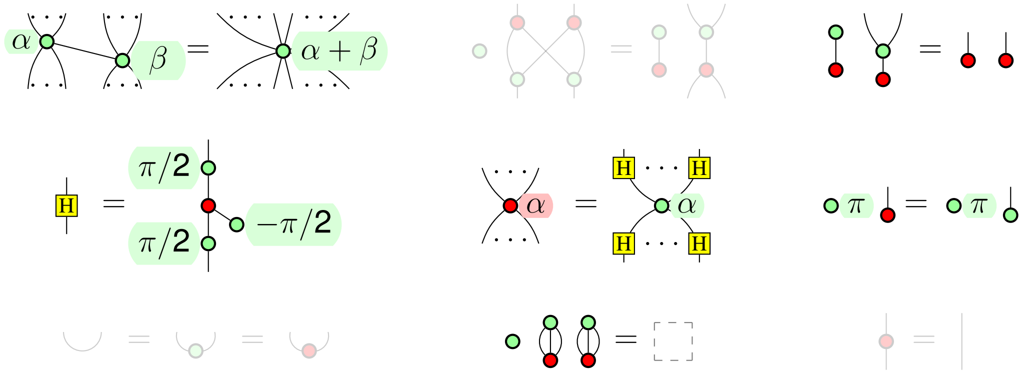


and



so at least one of them is necessary.

Summary of necessity arguments so far



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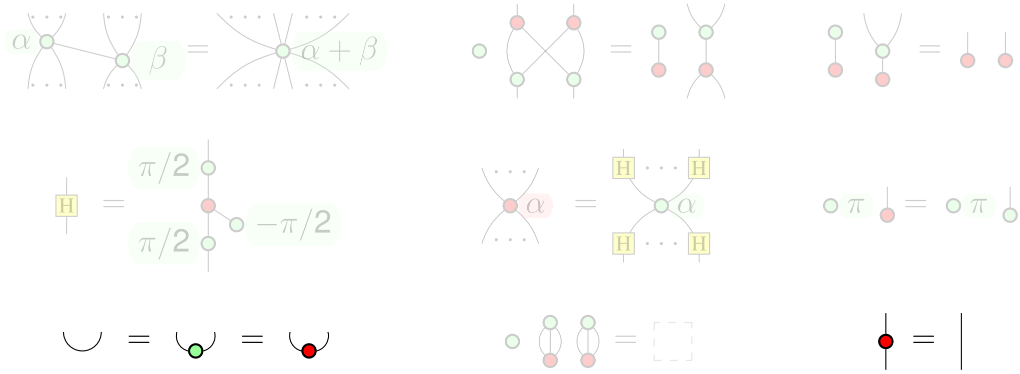

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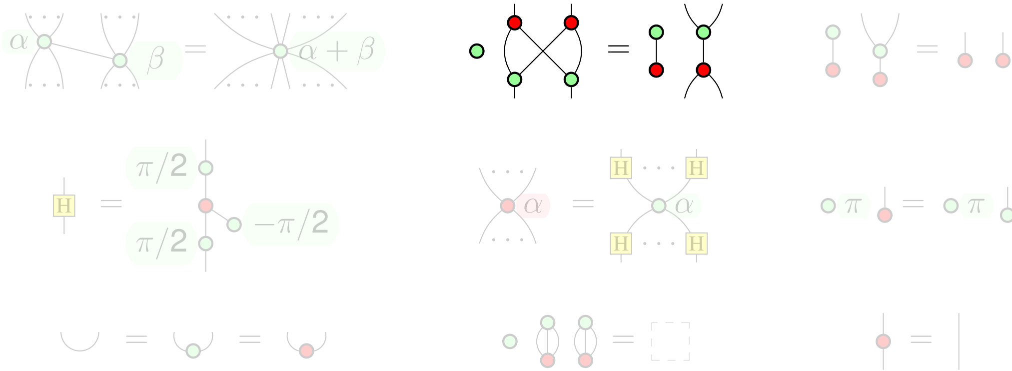


Working in a symmetric compact closed category.

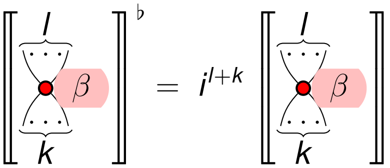
- spider rule, copy rule, Euler decomposition, colour change, zero rule, inverse rule are all necessary
- need at least one equality between a diagram containing a node and a diagram containing only a wire
- what about bialgebra?

Necessity of the bialgebra rule

Define alternative interpretation for ZX-calculus diagrams that acts like the usual interpretation on green spiders, wires, and the empty diagram, and adds phases to red spiders and Hadamard nodes as follows:

$$\left[\!\left[\, \boxed{H} \,\right]\!\right]^{\flat} = -i \left[\!\left[\, \boxed{H} \,\right]\!\right]$$

and



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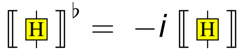

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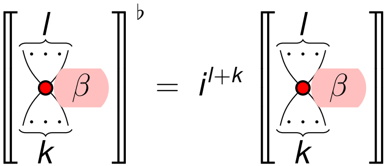
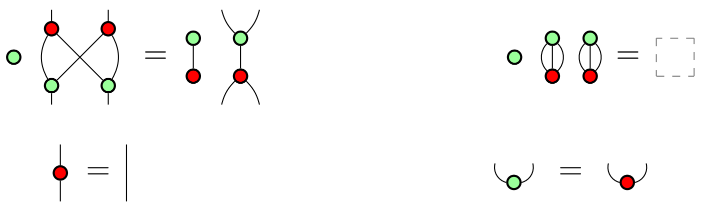


The following rules are not sound under the new interpretation, so at least one of them is necessary:



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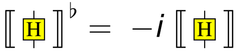

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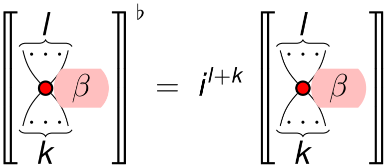
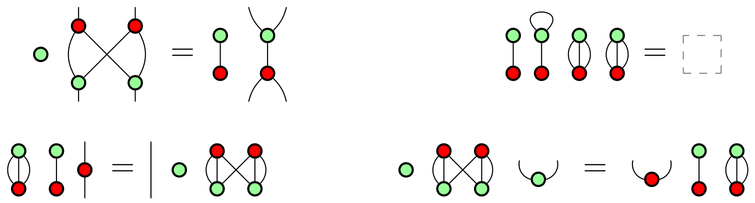


Can modify the rules so that bialgebra is necessary – but at the cost of introducing complicted scalars in some other rules:



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Conclusions

Conclusions and Outlook

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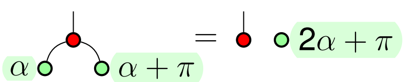
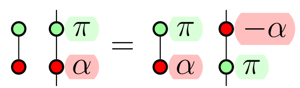
  - can show necessity of bialgebra only using complicated modification of rules
- additional rules are needed outside stabilizer QM, including supplementary rule [Perdrix & Wang, 2015] and (probably) π-commutation rule





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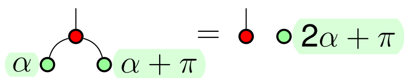
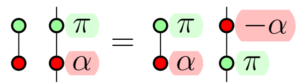
- have gone from 12 commonly-used rewrite rules (each in up to four versions) to just 9, while keeping completeness
- do not need to assume symmetry properties of spiders or Hadamard node
- some open questions concerning minimality remain:
  - of several rules relating spiders to categorical structure, unclear which (or how many) are necessary
  - can show necessity of bialgebra only using complicated modification of rules
- additional rules are needed outside stabilizer QM, including supplementary rule [Perdrix & Wang, 2015] and (probably) π-commutation rule





Thank you!