Quantum Foundations and Bell’s Theorem

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Outline

• Characteristics of Quantum Theory
• Local Realism and Bell’s Theorem
• Contextuality
• The Reality of the Wavefunction
Disclaimer

• I’m not in Foundations, Outsider’s impression
• Few technical details, except where simple
• Mainly to give a flavour of the issues in Quantum Foundations
• Highly incomplete (possible wrong in parts)
(Some) Questions in Quantum Foundations

• Meaning of the wavefunction?
• Meaning of measurement?
• One world or many?
• Real or not?
• Local or not?
• Difference between Classical and Quantum?
• Why is QM the way it is, not some other theory?
Quantum Theory in a Nutshell

- (Pure) state of a system represented by a vector in a complex Hilbert Space
- Observables represented by Hermitian operators
- Probabilistic outcomes of measurements
- State modified by measurement
- Heisenberg’s uncertainty leads to impossibility of simultaneous definite values for all properties
- Entanglement, non-locality
How Quantum is Different from Classical

• Classical theories
  – Allows definite (macro realistic) states of systems
  – Measurement just reveals state, noiseless in principle

• Quantum theory
  – Allows superposition of states
  – Distinct states may not be different (non-orthogonality)
  – Measurement intrinsically disturbing
Three strands to Foundations

• Looking for novel effects in quantum theory;
• Investigating conceptual issues in, and interpretations of, quantum theory; and
• Developing a deeper understanding of the structure of the theory (both mathematical and conceptual) for its own sake, for the purposes of finding a way to reconstruct the theory from more basic axioms, and for the purpose of going beyond quantum theory.
The Danger Zone: Interpretations

- Copenhagen (?)
- Many Worlds/Minds
- Shut up and calculate, non-interpretation
- Epistemic (states of knowledge)
- De Broglie-Bohm (non-local but realist)

We’ll ignore these issues here, save it for discussion over a pint
Two Main Approaches to Understanding QM

• Accept the classical world view
  – Find a way of interpreting/modifying quantum theory to fit, e.g. hidden variables.

• Accept quantum theory
  – Find a way by which the classical world emerges, e.g. decoherence programme
Einstein-Podolsky-Rosen (EPR 1935)

• Argued QM Incomplete
  – Probabilities of measurement outcomes due to ignorance of the actual underlying physical state
  – Appeared to sidestep Heisenberg’s Uncertainty
EPR

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

Position $x$
Momentum $p$

\[
[x, p] = i\hbar
\]

QM says a system cannot have simultaneous definite values for both $x$, $p$

\[
|\Psi\rangle_{AB} = \sum_x |x\rangle_A \otimes |x\rangle_B = \sum_p |p\rangle_A \otimes |-p\rangle_B
\]

• If Alice measures $x$, can predict Bob would have measured $x$ as well, therefore Bob must have had $x$ all along
• Conversely, if Bob measures $p$, he can predict Alice would have measured $-p$ as well, hence she must have had $-p$ all along
• Hence, they jointly could conclude that they both had particles with definite position and momentum all along, in contradiction with QM
EPR Summary

- **EPR assumptions**
  - Locality, Alice’s choice of measurement (position or momentum) does not influence the results of Bob’s measurement
  - Counterfactual reasoning, Alice concludes about the results of a measurement by Bob that isn’t performed, vice versa

- **EPR Concludes QM Incomplete.** The system of two particles are in a definite physical state. A complete physical theory should be able to describe the state in terms of definite outcomes of any possible set of measurements.
Bell’s Theorem

• How to test the “Classical Assumptions”?
  – Realism, underlying “hidden variables” that determine results of all measurements
  – Locality, the actions at one point cannot instantaneously influence the results at another

• Bell’s Theorem/Inequality
  – Takes the two assumptions above
  – Plus other “reasonable” assumptions
  – Finds an observable limit to such theories having these assumptions
  – QM “violates” this limit
Clauser-Horne-Shimony-Holt (CHSH)

Alice and Bob choose their measurements randomly and independently

Alice can measure property $A_1$ or $A_2$

Bob can measure property $B_1$ or $B_2$

Four possible sets of joint measurements:
$(A_1,B_1)$, $(A_1,B_2)$, $(A_2,B_1)$, $(A_2,B_2)$

Each measurement has two possible outcomes, $a,b = +1$ or $-1$

Correlation function for $(A_j,B_k)$

$$
\langle A_jB_k \rangle = (+1)P(a = +1, b = +1 | A_j, B_k) + (-1)P(a = -1, b = +1 | A_j, B_k)
+ (-1)P(a +1, b = -1 | A_j, B_k) + (+1)P(a = -1, b = -1 | A_j, B_k)
$$

CHSH Inequality

$$
|\langle A_1B_1 \rangle + \langle A_1B_2 \rangle + \langle A_2B_1 \rangle - \langle A_2B_2 \rangle| \leq 2
$$
Bell’s Theorem Example

Alice

+1

-1

Little or Big

Bob

+1

-1

Green or Red
Realism

One Sock

Λ = 1

λ = 1

λ = 2

λ = 3

λ = 4

λ = 5

λ = 6

λ = 7

λ = 8

λ = 9

λ = 10

λ = 11

λ = 12

λ = 13

λ = 14

λ = 15

λ = 16

Two Socks

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Science
Locality

- The outcome of Alice’s measurement does not depend on the choice of measurement by Bob.

- E.g. Bob’s decision to look at size or colour does not swap Alice’s sock.

- Alice’s sock is only pre-determined by $\lambda$. 
CSHS Inequality Cont.

\[ |\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| = S \]

Fix \( \lambda \). Assume definite values for \( A_1, A_2, B_1, B_2 \) exist simultaneously

\[ |\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle| = |A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2| \]
\[ = |A_1 (B_1 + B_2) + A_2 (B_1 - B_2)| \]
\[ = 2 \]

Alice’s choice does not affect Bob’s values

Any mixture of \( \lambda \) cannot increase this value.

For local realistic theories, \( S \leq 2 \)
QM and Local Realism

Example: Two maximally polarisation-entangled photons

\[ |S| \leq 2\sqrt{2} \]
Note on Loopholes

• Assumptions/loopholes
  – No post-selection, fair-sampling, high detection efficiency
  – Locality, measurements occur faster than light time of flight between Alice and Bob
  – Coincidence loophole
  – Independence of measurement settings
  – Memory loophole
  – Superdeterminism
Popescu-Rohrlich Boxes

\[ a \oplus b = xy \]

Non-signalling: Alice’s result does not say anything about Bob’s choice

\[ S = 4 \quad \text{Stronger non-locality than QM} \]

QM can output required function with \[ p = \frac{(2 + \sqrt{2})}{4} \approx 0.85 \]

Information Causality

- Alice wants Bob to have access to 2 bits of information but can only send 1
- With PR Boxes, Bob can independently decide which bit to retrieve

\[ x_0 \oplus x_1 \quad y = 0,1 \]

\[ a \quad x_0 \oplus a = m \quad b \]

\[ m \oplus b = x_y \]

- QRAC \( p = \cos^2 (\pi/8) \approx 0.85 \)
- In QM, m transmitted bits allows access to at most data set size m

GHZ(M) “Paradox”

- Three-party, “deterministic” counterexample to local realism

\[ |GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}} \left( |000\rangle + |111\rangle \right) \]

\[
\begin{align*}
\langle X \otimes Y \otimes Y \rangle &= -1 \\
\langle Y \otimes X \otimes Y \rangle &= -1 \\
\langle Y \otimes Y \otimes X \rangle &= -1 \\
\langle X \otimes X \otimes X \rangle &= +1
\end{align*}
\]

Local Realistic Model -1
(Non-)Contextuality

• Non-contextuality
  – All outcomes of measurements represent “elements of reality”
  – All observables defined for a QM system have definite values at all times
  – Underlying physical reality has definite outcomes regardless of configuration of measurements

• The non-commutivity of QM results in contextuality in higher than 3 dimensions

Sketch of Non-contextual assignment of projection outcomes for a qubit
(Bell-)Kochen-Specker Theorem

- In Hilbert space of dimension 3, 117 projections cannot simultaneously be ascribed definite outcomes consistently.
- Easier proof in 4 dimensions (Cabello et al 1997)

$$P_j = \frac{|u_j\rangle\langle u_j|}{\langle u_j \mid u_j \rangle}$$

$$1 = P_1 + P_2 + P_3 + P_4$$

Impossible to only assign a single 1 and three 0s to each column consistently.

Trivial proof, odd versus even.
Contextuality and Bell

- Bell Non-Locality a form of Contextuality
- Locality imposes contextual constraint
Reality of the Wavefunction

• Ontic
  – Wavefunction is “real”
  – Wavefunction represents the physical state

• Epistemic
  – Wavefunction is a “state of knowledge”
  – Exists deeper layer of physical reality, wavefunction is a statistical description
Is the wavefunction real?

- $\Psi$-Epistemic: State of knowledge. The same actual physical state could be part of the ensembles for two different wavefunctions. “Collapse”=Bayesian Update.
- $\Psi$-Ontic: Real in the sense that different wavefunctions represent different underlying physical configurations.

**Epistemic vs Ontic**

Disjoint distributions means $L$ constitutes a physical property, i.e. $\lambda$ determines $L$

Non-disjoint distributions means $L$ does not uniquely define collection of $\lambda$
Epistemic Approaches

• Reproduce “quantum” features from underlying epistemic toy models
  – E.g. Spekkens Toy Model

• Cannot reproduce all quantum phenomena
  – E.g. Bell violations, BKS
Pusey-Barrett-Rudolph (PBR)

- Under some “natural assumptions”, wavefunction cannot be interpreted statistically
  - There exists a real physical state, objective and independent of observer
  - Systems can be prepared independently

\[ |\psi_0\rangle = |0\rangle \]
\[ |\psi_1\rangle = |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \]

\[ \langle \psi_0 | \psi_1 \rangle = \frac{1}{\sqrt{2}} \]

Epistemic view: overlap in actual underlying distribution of states

\[ \Delta \neq \emptyset \]

Independently prepare
\[ |\psi_j\rangle \otimes |\psi_k\rangle \]

\[ |\xi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \]
\[ |\xi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |-\rangle + |1\rangle \otimes |+\rangle) \]

Each outcome orthogonal to one of the possible input states

\[ |\xi_3\rangle = \frac{1}{\sqrt{2}} (|+\rangle \otimes |1\rangle + |-\rangle \otimes |0\rangle) \]
\[ |\xi_4\rangle = \frac{1}{\sqrt{2}} (|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle) \]

Requires no overlap, otherwise potential for confusion and getting “wrong result”

The quantum state cannot be interpreted statistically, Matthew F. Pusey, Jonathan Barrett & Terry Rudolph, arXiv:1111.3328v1
PBR Cont.

• Theorem holds in presence of imperfections and noise
• Can generalize to any pair of non-orthogonal quantum states
• Hence any underlying $\mu_{\psi}(\lambda)$ must be disjoint for all pairs of wavefunctions
• Hence different wavefunctions constitute distinct physical properties, are ontic
• Dropping “Preparation Independence” allows epistemic interpretation that matched QM

The quantum state can be interpreted statistically, P. G. Lewis, D. Jennings, J. Barrett, T. Rudolph, arXiv:1201.6554v1
Undiscussed

- Hardy’s Paradox
- Leggett Inequalities
- Leggett-Garg Inequalities
- Multi-partite non-locality
- Uncertainty bounds
- Generalized probability theories
- Decoherence Programme
- “Reasonable Axioms” implying QM
- Relativistic QM
- QM and Gravity
- Etc...