In search of the spectrum

Tom Leinster
University of Edinburgh
Meanings of ‘spectrum’

Physics & chemistry: emission/absorption spectra, mass spectroscopy, etc.

Linear algebra: eigenvalues of an operator, with their algebraic multiplicities

Functional analysis and PDEs: more sophisticated relatives of the linear algebra notion

Commutative algebra: the spectrum of a commutative ring is its space of prime ideals

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Homotopy theory: spectrum ≈ infinite loop space.

Non-quantum computing:
What ‘spectrum’ means in this talk

I will use ‘spectrum’ to mean the set $\text{Spec}(T)$ of eigenvalues of a linear operator $T$ on a finite-dimensional vector space, with their algebraic multiplicities. Algebraically and categorically, its properties seem awkward. For instance, given operators $S$ and $T$ on the same space, knowing $\text{Spec}(S)$ and $\text{Spec}(T)$ tells you almost nothing about $\text{Spec}(S \circ T)$ or $\text{Spec}(S + T)$. But socially, the spectrum is important! So, there ought to be a clean abstract characterization of it. This talk offers one.
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The goal of the talk is to explain this theorem:

**Theorem**

Among all invariants of linear operators on finite-dimensional vector spaces, the universal cyclic, balanced invariant is the set of nonzero eigenvalues with their algebraic multiplicities.

**Plan:**
1. Linear Algebra Done Right
2. Invariants
3. Cyclic invariants
4. Balanced invariants
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1. Linear Algebra Done Right
Linear Algebra Done Right
by Sheldon Axler

Sheldon Axler (1975)

Book (1996)
The eventual image and eventual kernel
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The eventual image $\text{im}^\infty(T)$ of $T$ is the intersection of the chain of subspaces

$$\text{im}(T) \supseteq \text{im}(T^2) \supseteq \text{im}(T^3) \supseteq \ldots.$$
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The eventual kernel $\text{ker}^\infty(T)$ of $T$ is the union of the chain of subspaces

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- let \( k \) be an algebraically closed field
- let \( X \) be a finite-dimensional vector space over \( k \)
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The **eventual image** \( \text{im}^\infty(T) \) of \( T \) is the intersection of the chain of subspaces

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\text{im}(T) \supseteq \text{im}(T^2) \supseteq \text{im}(T^3) \supseteq \cdots.
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This is *smaller* than the ordinary image \( \text{im}(T) \).

The **eventual kernel** \( \text{ker}^\infty(T) \) of \( T \) is the union of the chain of subspaces

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\text{ker}(T) \subseteq \text{ker}(T^2) \subseteq \text{ker}(T^3) \subseteq \cdots.
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This is *larger* than the ordinary kernel \( \text{ker}(T) \).
The first canonical decomposition of a linear operator

Recall the notation: $T$ is a linear operator on a fin-dim vector space $X$. Lemma

$$X/\ker(T) = \ker(T^\infty)/\ker(T) \oplus \im(T^\infty)/\ker(T)$$

where $T^0$ is nilpotent (i.e. some power of $T^0$ is 0) and $T^\times$ is invertible.

In other words: (i) $X = \ker(T^\infty) \oplus \im(T^\infty)$, and (ii) $T$ restricts to a nilpotent operator on $\ker(T^\infty)$ and an invertible operator on $\im(T^\infty)$.

This is the unique decomposition of $T$ as the direct sum of a nilpotent operator and an invertible operator.

Contrast: usually $X \neq \ker(T) + \im(T)$, although we do have $\dim X = \dim \ker(T) + \dim \im(T)$. 
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$X \oplus T = \ker^\infty(T) \oplus T_0 \oplus \text{im}^\infty(T) \oplus T^\times$
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**Contrast:** usually $X \neq \ker(T) + \operatorname{im}(T)$, although we do have $\dim X = \dim \ker(T) + \dim \operatorname{im}(T)$. 
The second canonical decomposition of a linear operator

\[ \frac{X}{\ker(T - \lambda)} = \bigoplus_{\lambda \in k} \frac{\ker(\infty(T - \lambda))}{\ker(T - \lambda)} \]

where \( T - \lambda \) is nilpotent (i.e. the only eigenvalue of \( T - \lambda \) is \( \lambda \)).

We call \( \ker(\infty(T - \lambda)) \) the eventual eigenspace (or generalized eigenspace) with value \( \lambda \).

It is bigger than the ordinary eigenspace, which consists of those vectors annihilated by applying \( T - \lambda \) just once.

Contrast: usually \( X \neq \bigoplus_{\lambda \in k} \ker(T - \lambda) \). They are equal iff \( T \) is diagonalizable.

The eventual eigenspace \( \ker(\infty(T - \lambda)) \) is trivial for all except finitely many values of \( \lambda \in k \)—namely, the eigenvalues.
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Theorem

\[ X \circ T = \bigoplus_{\lambda \in k} \ker^\infty (T - \lambda) \circ T_\lambda \]
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\[ X \otimes T = \bigoplus_{\lambda \in k} \ker^\infty (T - \lambda) \otimes T_\lambda \]

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\[ X \cong T = \bigoplus_{\lambda \in \mathbb{K}} \ker^\infty (T - \lambda) \cong T_\lambda \]

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Comparing the two decompositions

First decomposition:

\[ X/\cup_\infty T = \ker\infty(T)/\cup_\infty T_0 \oplus \im\infty(T)/\cup_\infty T \times \]

where \( T_0 \) is nilpotent and \( T \times \) is invertible.

Second decomposition:

\[ X/\cup_\infty T = \lambda \in k \ker\infty(T-\lambda)/\cup_\infty T_\lambda \]

where \( T_\lambda - \lambda \) is nilpotent.

The two operators called ‘\( T_0 \)’ are the same, and the second decomposition refines the first:

\[ \im\infty(T)/\cup_\infty T \times = \lambda \neq 0 \ker\infty(T-\lambda)/\cup_\infty T_\lambda . \]
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\[ X \odot T = \ker^\infty(T) \odot T_0 \oplus \im^\infty(T) \odot T^x \]

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\[ \im^\infty ( T ) \odot T^\times = \bigoplus_{\lambda \neq 0} \ker^\infty ( T - \lambda ) \odot T_\lambda . \]
Algebraic multiplicity

Let $\lambda \in k$. The algebraic multiplicity of $\lambda$ in $T$ is $\alpha_T(\lambda) = \dim \ker \infty(T - \lambda)$. (We could also call it the 'dynamic multiplicity'.)

Note that $X = \lambda \in k \ker \infty(T - \lambda) \Rightarrow \dim X = \sum \lambda \in k \alpha_T(\lambda)$.

We can then define:

- **Trace**: $\operatorname{tr}(T) = \sum \lambda \in k \alpha_T(\lambda) \cdot \lambda$
- **Determinant**: $\det(T) = \prod \lambda \in k \lambda \alpha_T(\lambda)$
- **Characteristic polynomial**: $\chi_T(x) = \prod \lambda \in k (x - \lambda) \alpha_T(\lambda) = \det(x - T)$.

All have their usual meanings!
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Functoriality of the eventual image

Given a category $C$, let $\text{Endo}(C)$ denote the category whose:

- objects are endomorphisms $X/\cong T$ in $C$
- maps $X/\cong T/\to Y/\cong S$ are maps $f : X/\to Y$ in $C$ such that $S \circ f = f \circ T$.

Let $\text{FDVect}$ be the category of finite-dimensional vector spaces. We're interested in $\text{Endo}(\text{FDVect})$, the category of linear operators.

There is a functor $\text{Endo}(\text{FDVect})/\to \text{Endo}(\text{FDVect}) X/\cong \text{im}_\infty(T)/\cong T \times/\to \text{im}_\infty(S)/\cong S \times$.

On maps, it's defined by restriction: any map of operators $f : X/\to Y$ restricts to a map $f : \text{im}_\infty(T)/\to \text{im}_\infty(S)$.
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- maps $X/T \to Y/S$ are maps $f : X \to Y$ in $\mathcal{C}$ such that $S \circ f = f \circ T$.

Let $\text{FDVect}$ be the category of finite-dimensional vector spaces. We're interested in $\text{Endo}(\text{FDVect})$, the category of linear operators. There is a functor $\text{Endo}(\text{FDVect}) \to \text{Endo}(\text{FDVect}) \times \text{im}_\infty(T) \times \text{im}_\infty(S) \times$. On maps, it's defined by restriction: any map of operators $f : X/T \to Y/S$ restricts to a map $f : \text{im}_\infty(T) \times \to \text{im}_\infty(S) \times$. 
Functoriality of the eventual image

Given a category $\mathcal{C}$, let $\text{Endo}(\mathcal{C})$ denote the category whose:

- objects are endomorphisms $X \bowtie T$ in $\mathcal{C}$
Functoriality of the eventual image

Given a category $\mathcal{C}$, let $\text{Endo}(\mathcal{C})$ denote the category whose:

- objects are endomorphisms $X \circ T$ in $\mathcal{C}$
- maps $X \circ T \to Y \circ S$ are maps $f: X \to Y$ in $\mathcal{C}$ such that $S \circ f = f \circ T$. 

Let $\text{FDVect}$ be the category of finite-dimensional vector spaces. We're interested in $\text{Endo}(\text{FDVect})$, the category of linear operators.

There is a functor $\text{Endo}(\text{FDVect}) \to \text{Endo}(\text{FDVect})$ such that:

- objects are sent to objects
- maps $X \circ T \to Y \circ S$ restrict to maps $f: \text{im} \infty(T) \to \text{im} \infty(S)$. 

On maps, it's defined by restriction: any map of operators $f: X \to Y$ restricts to a map $f: \text{im} \infty(T) \to \text{im} \infty(S)$. 

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Functoriality of the eventual image

Given a category $\mathcal{C}$, let $\text{Endo}(\mathcal{C})$ denote the category whose:

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We’re interested in $\text{Endo}(\text{FDVect})$, the category of linear operators.
Functoriality of the eventual image

Given a category $\mathcal{C}$, let $\text{Endo}(\mathcal{C})$ denote the category whose:

- **objects** are endomorphisms $X\mathrel{\overset{\mathcal{O}}\rightarrow} T$ in $\mathcal{C}$
- **maps** $X\mathrel{\overset{\mathcal{O}}\rightarrow} T \rightarrow Y\mathrel{\overset{\mathcal{O}}\rightarrow} S$ are maps $f: X \rightarrow Y$ in $\mathcal{C}$ such that $S \circ f = f \circ T$.

Let $\mathbf{FDVect}$ be the category of finite-dimensional vector spaces.
We’re interested in $\text{Endo}(\mathbf{FDVect})$, the category of linear operators.
There is a functor

$$
\begin{align*}
\text{Endo}(\mathbf{FDVect}) & \quad \longrightarrow \quad \text{Endo}(\mathbf{FDVect}) \\
X\mathrel{\overset{\mathcal{O}}\rightarrow} T & \quad \mapsto \quad \text{im}^{\infty}(T)\mathrel{\overset{\mathcal{O}}\rightarrow} T^\times.
\end{align*}
$$
Functoriality of the eventual image

Given a category $\mathcal{C}$, let $\text{Endo}(\mathcal{C})$ denote the category whose:

- objects are endomorphisms $X \circled{T}$ in $\mathcal{C}$
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Let $\text{FDVect}$ be the category of finite-dimensional vector spaces.

We’re interested in $\text{Endo}(\text{FDVect})$, the category of linear operators.

There is a functor

$$\text{Endo}(\text{FDVect}) \xrightarrow{\text{ }} \text{Endo}(\text{FDVect})$$

$X \circled{T} \xmapsto{\text{restriction}} \text{im}^{\infty}(T) \circled{T^\times}$.

On maps, it’s defined by restriction:
Given a category $\mathcal{C}$, let $\text{Endo}(\mathcal{C})$ denote the category whose:

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\end{align*}
$$

On maps, it’s defined by restriction: any map of operators $f: X \circ T \rightarrow Y \circ S$ restricts to a map $f: \text{im}^\infty(T) \circ T^\times \rightarrow \text{im}^\infty(S) \circ S^\times$. 

Functoriality of the eventual image
2. Invariants
Definition and examples

Let $E$ be a category. An invariant of objects of $E$ is a function 
$\text{ob}(E) / \cong = \{\text{isomorphism classes of objects of } E\} \rightarrow \Omega$, 
where $\Omega$ is some set.

We're studying invariants of linear operators. So take $E = \text{Endo}(\text{FDVect})$.

Examples of invariants of linear operators $X$:

- The trace or determinant or characteristic polynomial.
- The algebraic multiplicity $\alpha_T$ (33 etc.).
- The spectrum $\text{Spec}(T)$, defined as the set of eigenvalues with their algebraic multiplicities. This is a finite subset-with-multiplicities of $k$.
- The invertible spectrum $\text{Spec} \times (T)$, defined as the set of nonzero eigenvalues with their algebraic multiplicities. This is a finite subset-with-multiplicities of $k \times \{0\}$.
- The isomorphism type of $\text{im} \infty (T) \times T$. (Can describe these iso types concretely via Jordan normal form.)
Definition and examples

Let $\mathcal{E}$ be a category. An invariant of objects of $\mathcal{E}$ is a function

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Examples of invariants of linear operators $X/\cong$:  

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Definition and examples

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**Examples of invariants of linear operators $X/\sim$:**

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Definition and examples

Let $\mathcal{E}$ be a category. An invariant of objects of $\mathcal{E}$ is a function

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Examples of invariants of linear operators $X \otimes T$:

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Definition and examples

Let $\mathcal{E}$ be a category. An invariant of objects of $\mathcal{E}$ is a function

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**Examples** of invariants of linear operators $X \otimes T$:

- The trace or determinant or characteristic polynomial.
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Definition and examples

Let $\mathscr{C}$ be a category. An invariant of objects of $\mathscr{C}$ is a function

$$\text{ob}(\mathscr{C})/\cong = \{\text{isomorphism classes of objects of } \mathscr{C}\} \longrightarrow \Omega,$$

where $\Omega$ is some set.

We’re studying invariants of linear operators. So take $\mathscr{C} = \text{Endo}(\text{FDVect})$.

Examples of invariants of linear operators $X \mathrel{\circ} T$:

- The trace or determinant or characteristic polynomial.
- The algebraic multiplicity $\alpha_T(33)$ (etc.).
- The spectrum $\text{Spec}(T)$, defined as the set of eigenvalues with their algebraic multiplicities. This is a finite subset-with-multiplicities of $k$.
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- The isomorphism type of $\text{im}^\infty(T) \mathrel{\circ} T^\times$. 
Definition and examples

Let \( \mathcal{E} \) be a category. An invariant of objects of \( \mathcal{E} \) is a function

\[
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\]

where \( \Omega \) is some set.

We’re studying invariants of linear operators. So take \( \mathcal{E} = \text{Endo} (\text{FDVect}) \).

Examples of invariants of linear operators \( X \otimes T \):

- The trace or determinant or characteristic polynomial.
- The algebraic multiplicity \( \alpha_T(33) \) (etc.).
- The spectrum \( \text{Spec}(T) \), defined as the set of eigenvalues with their algebraic multiplicities. This is a finite subset-with-multiplicities of \( k \).
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- The isomorphism type of \( \text{im}^\infty(T) \circ T^\times \).

(Or these iso types concretely via Jordan normal form.)
Digression on the invertible spectrum

We just defined the invertible spectrum $\text{Spec} \times (T)$ to be the set-with-multiplicities of nonzero eigenvalues. There's also $\text{Spec} (T)$, the set-with-multiplicities of all eigenvalues. Suppose we know $\dim X$. Then knowing $\text{Spec} \times (T)$ is equivalent to knowing $\text{Spec} (T)$, because

$$\alpha_0 (T) = \dim X - \sum_{\lambda \neq 0} \alpha_\lambda (T).$$

Why are the nonzero eigenvalues interesting?

- because they tell you the values of $\mu$ for which $\mu T - I$ is singular
- because of cyclicity.
Digression on the invertible spectrum

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Digression on the invertible spectrum

We just defined the invertible spectrum Spec$^\times (T)$ to be the set-with-multiplicities of nonzero eigenvalues.

There's also Spec($T$), the set-with-multiplicities of all eigenvalues.

Suppose we know dim $X$. 
Digression on the invertible spectrum

We just defined the invertible spectrum \( \text{Spec}^\times(T) \) to be the set-with-multiplicities of nonzero eigenvalues.

There's also \( \text{Spec}(T) \), the set-with-multiplicities of all eigenvalues.

Suppose we know \( \dim X \).

Then knowing \( \text{Spec}^\times(T) \) is equivalent to knowing \( \text{Spec}(T) \).

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Why are the **nonzero** eigenvalues interesting?

- because they tell you the values of $\mu$ for which $\mu T - I$ is singular
- because of cyclicity...
3. *Cyclic invariants*
Definition

Let \( C \) be a category. An invariant \( \Phi \) of endomorphisms in \( C \) is cyclic if
\[
\Phi(g \circ f) = \Phi(f \circ g)
\]
whenever \( X \xrightarrow{f} Y \xleftarrow{g} \) in \( C \).

Example: Trace is cyclic: \( \text{tr}(g \circ f) = \text{tr}(f \circ g) \).

A cyclic invariant \( \Phi \) assigns a value to any cycle
\[
X_n \xrightarrow{f_n} X_1 \xleftarrow{f_1} X_2 \xrightarrow{f_2} \ldots
\]
in \( C \), since \( \Phi(f_i \circ/ \ldots \circ/ f_1 \circ/ \ldots \circ/ f_n \circ/ \ldots) \) is independent of \( i \).
Definition

Let $\mathcal{C}$ be a category. An invariant $\Phi$ of endomorphisms in $\mathcal{C}$ is cyclic if

$$\Phi(g \circ f) = \Phi(f \circ g)$$

whenever $X \xleftarrow{g} \xrightarrow{f} Y$ in $\mathcal{C}$. 

Example: Trace is cyclic: $\text{tr}(g \circ f) = \text{tr}(f \circ g)$. 

A cyclic invariant $\Phi$ assigns a value to any cycle $X_1 \xrightarrow{f} X_2 \xrightarrow{f} \ldots \xrightarrow{f} X_n \xleftarrow{g} Y$ in $\mathcal{C}$, since $\Phi(f_i \circ f_{i+1}\circ/\ldots\circ f_1 \circ f_n\circ/\ldots\circ f_{i-1})$ is independent of $i$. 
**Definition**

Let $\mathcal{C}$ be a category. An invariant $\Phi$ of endomorphisms in $\mathcal{C}$ is cyclic if

$$\Phi(g \circ f) = \Phi(f \circ g)$$

whenever $X \xrightarrow{f} Y$ in $\mathcal{C}$.

**Example:** Trace is cyclic: $\text{tr}(g \circ f) = \text{tr}(f \circ g)$. 
Definition

Let \( \mathcal{C} \) be a category. An invariant \( \Phi \) of endomorphisms in \( \mathcal{C} \) is cyclic if

\[
\Phi(g \circ f) = \Phi(f \circ g)
\]

whenever \( X \xrightarrow{f} Y \) in \( \mathcal{C} \).

Example: Trace is cyclic: \( \text{tr}(g \circ f) = \text{tr}(f \circ g) \).

A cyclic invariant \( \Phi \) assigns a value to any cycle

\[
\begin{array}{c}
X_n \xrightarrow{f_n} X_1 \xrightarrow{f_1} X_2 \\
\downarrow \quad \quad \downarrow \\
X_3 \\
\end{array}
\]

in \( \mathcal{C} \), since \( \Phi(f_i \circ \cdots \circ f_1 \circ f_n \circ \cdots f_{i+1}) \) is independent of \( i \).
The eventual image is a cyclic invariant
The eventual image is a cyclic invariant

Given

\[ X \xleftarrow{f} Y \]

in \textit{FDVect}

Conclusion: The isom'm type of $\text{im}_\infty(T)/\text{uni}_T \times T$ is a cyclic invariant of $X/\text{uni}_T$. (In fact, this is the initial cyclic invariant of linear operators.)
The eventual image is a cyclic invariant

Given

\[ X \xleftarrow{f} Y \xrightarrow{g} \]

in \textbf{FDVect}, we get

\[ X \otimes gf \xleftarrow{g} Y \otimes fg \xrightarrow{f} \]

in \textbf{Endo(FDVect)}

Conclusion: The isom'm type of \( \text{im} \infty \left( gf \right) \xrightarrow{uni} \left( gf \right) \times f \) is a cyclic invariant of \( X \xrightarrow{uni} Y \times g \).

(In fact, this is the initial cyclic invariant of linear operators.)
The eventual image is a cyclic invariant

Given

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow{g} \\
\end{array}
\]

in \textbf{FDVect}, we get

\[
\begin{array}{ccc}
X & \xleftarrow{gf} & Y \\
\downarrow{g} & & \downarrow{g} \\
\end{array}
\]

in \textbf{Endo(FDVect)}, hence by functoriality of the eventual image,

\[
\begin{array}{ccc}
\text{im}^\infty (gf) & \xleftarrow{(gf)^\times} & \text{im}^\infty (fg) \\
\downarrow{g} & & \downarrow{g} \\
\end{array}
\]

in \textbf{Endo(FDVect)}.

Conclusion: The isomorphism type of \(\text{im}^\infty (T)\) is a cyclic invariant of \(X\).

(In fact, this is the initial cyclic invariant of linear operators.)
The eventual image is a cyclic invariant

Given

\[
\begin{array}{ccc}
X & \xleftarrow{f} & Y \\
\xrightarrow{g} & \quad & \\
\end{array}
\]

in \textbf{FDVect}, we get

\[
\begin{array}{ccc}
X \xleftarrow{gf} & \xrightarrow{f} & Y \xleftarrow{fg} \\
\xrightarrow{g} & \quad & \\
\end{array}
\]

in \textbf{Endo(FDVect)}, hence by functoriality of the eventual image,

\[
\begin{array}{ccc}
\text{im}^\infty(gf) \xleftarrow{(gf)^\times} & \xrightarrow{f} & \text{im}^\infty(fg) \xleftarrow{(fg)^\times} \\
\xrightarrow{g} & \quad & \\
\end{array}
\]

in \textbf{Endo(FD Vect)}.

But the composites of \( \xrightarrow{f} \) are \((gf)^\times\) and \((fg)^\times\)
The eventual image is a cyclic invariant

Given

\[
\begin{array}{c}
X \xrightarrow{f} Y \\
\xleftarrow{g} & \\
\end{array}
\]

in \textbf{FDVect}, we get

\[
\begin{array}{c}
X \circ gf \xrightarrow{f} Y \circ fg \\
\xleftarrow{g} & \\
\end{array}
\]

in \textbf{Endo(FDVect)}, hence by functoriality of the eventual image,

\[
\begin{array}{c}
im^\infty (gf) \circ (gf)^\times \xrightarrow{f} \im^\infty (fg) \circ (fg)^\times \\
\xleftarrow{g} & \\
\end{array}
\]

in \textbf{Endo(FDVect)}.

But the composites of \[
\begin{array}{c}
\xleftarrow{f} & \\
\xrightarrow{g} & \\
\end{array}
\]
are \((gf)^\times\) and \((fg)^\times\), which are invertible.
The eventual image is a cyclic invariant

Given

\[
\begin{array}{ccc}
X & \xleftarrow{f} & Y \\
\xleftarrow{g} & \end{array}
\]

in \textbf{FDVect}, we get

\[
\begin{array}{ccc}
X \circ gf & \xleftarrow{f} & Y \circ fg \\
\xleftarrow{g} & \end{array}
\]

in \textbf{Endo(FDVect)}, hence by functoriality of the eventual image,

\[
\begin{array}{ccc}
\text{im}^\infty (gf) \circ (gf)^\times & \xleftarrow{f} & \text{im}^\infty (fg) \circ (fg)^\times \\
\xleftarrow{g} & \end{array}
\]

in \textbf{Endo(FDVect)}.

But the composites of \( \xleftarrow{f} \) are \((gf)^\times\) and \((fg)^\times\), which are invertible, so

\[
\text{im}^\infty (gf) \circ (gf)^\times \cong \text{im}^\infty (fg) \circ (fg)^\times.
\]
The eventual image is a cyclic invariant

Given

\[ \begin{array}{ccc}
X & \xleftarrow{f} & Y \\
\xleftarrow{g} & & \\
\end{array} \]

in \textbf{FDVect}, we get

\[ \begin{array}{ccc}
X \ominus gf & \xleftarrow{f} & Y \ominus fg \\
\xleftarrow{g} & & \\
\end{array} \]

in \textbf{Endo}(\textbf{FDVect}), hence by functoriality of the eventual image,

\[ \begin{array}{ccc}
im^\infty (gf) \ominus (gf)^\times & \xleftarrow{f} & im^\infty (fg) \ominus (fg)^\times \\
\xleftarrow{g} & & \\
\end{array} \]

in \textbf{Endo}(\textbf{FDVect}).

But the composites of \( \xleftarrow{f} \xrightarrow{g} \) are \((gf)^\times\) and \((fg)^\times\), which are invertible, so

\[ \begin{array}{ccc}
im^\infty (gf) \ominus (gf)^\times & \cong & im^\infty (fg) \ominus (fg)^\times \\
\xleftarrow{f} \xrightarrow{g} & & \\
\end{array} \]
The eventual image is a cyclic invariant

Given

\[
X \xrightarrow{f} Y
\]

in \text{FDVect}, we get

\[
X \circlearrowleft g f \xleftarrow{g} X \circlearrowright \circlearrowleft g f \xleftrightarrow{g} Y \circlearrowright f g \xleftarrow{g}
\]

in \text{Endo(FDVect)}, hence by functoriality of the eventual image,

\[
\im^{\infty}(gf) \circlearrowleft (gf)^{\times} \xleftrightarrow{g} \im^{\infty}(fg) \circlearrowright (fg)^{\times}
\]

in \text{Endo(FDVect)}.

But the composites of \( g f \) are \((gf)^{\times}\) and \((fg)^{\times}\), which are invertible, so

\[
\im^{\infty}(gf) \circlearrowleft (gf)^{\times} \cong \im^{\infty}(fg) \circlearrowright (fg)^{\times}
\]

Conclusion: The isom’m type of \( \im^{\infty}(T) \circlearrowleft T^{\times} \) is a cyclic invariant of \( X \circlearrowleft T \).
The eventual image is a cyclic invariant

Given

\[ X \xleftarrow{f} Y \]

in \textbf{FDVect}, we get

\[ X \rightleftharpoons g \quad f \quad Y \rightleftharpoons g \]

in \textbf{Endo(FDVect)}, hence by functoriality of the eventual image,

\[ \text{im}^\infty (gf) \rightleftharpoons (gf)^\times \quad f \quad \text{im}^\infty (fg) \rightleftharpoons (fg)^\times \]

in \textbf{Endo(FDVect)}.

But the composites of \( \xleftarrow{f} g \) are \((gf)^\times\) and \((fg)^\times\), which are invertible, so

\[ \text{im}^\infty (gf) \rightleftharpoons (gf)^\times \cong \text{im}^\infty (fg) \rightleftharpoons (fg)^\times \]

Conclusion: \textit{The isom’m type of im}^\infty (T) \rightleftharpoons T^\times \textit{ is a cyclic invariant of } X \rightleftharpoons T. \]

(In fact, this is the \textit{initial} cyclic invariant of linear operators.)
The invertible spectrum is a cyclic invariant.
The invertible spectrum is a cyclic invariant

Again, take \( X \xleftarrow{g} Y \xrightarrow{f} \) in \( \text{FDVect} \).

A similar argument shows that \( \ker \infty (gf - \lambda) \cong \ker \infty (fg - \lambda) \) for all \( \lambda \neq 0 \).

So \( gf \) and \( fg \) have the same nonzero eigenvalues with the same algebraic multiplicities.

Conclusion: The invertible spectrum \( \text{Spec} \times \) is a cyclic invariant.

Fact of life: This fails for the eigenvalue 0. E.g. consider the first inclusion and first projection \( k \xleftrightarrow{g} k \oplus k \).

One composite has 0 as an eigenvalue, and the other does not. So, the multiplicity of 0 as an eigenvalue is not a cyclic invariant.
The invertible spectrum is a cyclic invariant

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4. Balanced invariants
Suppose someone tells you ‘left perfect’ is an important property of rings. Then you know there must be an equally important invariant, ‘right perfect’.

Formally: given a ring $R$, define $R^{\text{op}}$ by reversing the order of multiplication. This defines an automorphism $(\_\_)^{\text{op}}$ of the category of rings.

Then $R$ is right perfect $\iff R^{\text{op}}$ is left perfect.

Given a compact metric space $X$, let $N^1(X)$ be the number of balls of radius 1 needed to cover $X$.

If $N^1(X)$ is interesting then so too must be $N^r(X)$ (the number of balls of radius $r$ needed to cover $X$), for every $r > 0$.

Formally: define $tX$ to be $X$ scaled up by a factor of $t$.

For each $t > 0$, this defines an automorphism $t\cdot -$ of the category of metric spaces.

Then $N^r(X) = N^1\left(\frac{1}{r}X\right)$. 
Looking at an object from all directions

Suppose someone tells you ‘left perfect’ is an important property of rings.
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We agree that being invertible is an important property of linear operators.
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So, given an operator $T$, it must be important to ask for each $\lambda \in k$ whether $T - \lambda$ is invertible.
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So, given an operator \( T \), it must be important to ask for each \( \lambda \in k \) whether \( T - \lambda \) is invertible...and of course it is important!
 Balanced invariants

Let $\mathcal{E}$ be a category. An invariant $\Phi$ of objects of $\mathcal{E}$ is balanced if it ‘looks at objects from all directions’.

E.g. \{rings\} \leftarrow true, false is not balanced, but \{rings\} \leftarrow (left perf?, right perf?) \rightarrow \{true, false\} \times \{true, false\} is.

E.g. \{compact metric spaces\} \rightarrow \mathbb{N} is not balanced, but \{compact metric spaces\} \rightarrow (\mathbb{N}_r)_{r > 0} \rightarrow \{functions \mathbb{R}_+ \rightarrow \mathbb{N}\} is.

E.g. \{linear operators\} \leftarrow injective? \rightarrow \{true, false\} is not balanced, but \{linear operators\} \leftarrow eigenvalues \rightarrow \{subsets of k\} is.
Balanced invariants

Let $\mathcal{E}$ be a category. An invariant $\Phi$ of objects of $\mathcal{E}$ is balanced if it ‘looks at objects from all directions’: formally, for every automorphism $F$ of $\mathcal{E}$,

$$\Phi(E_1) = \Phi(E_2) \implies \Phi(F(E_1)) = \Phi(F(E_2))$$

for $E_1, E_2 \in \mathcal{E}$.
Balanced invariants

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In other words: $\Phi$ is balanced if $\Phi(E)$ determines $\Phi(F(E))$. 
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$\{\text{compact metric spaces}\} \xrightarrow{N_{r > 0}} \{\text{functions} \ R^+ \to N\}$ is.

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Balanced invariants

Let \( \mathcal{E} \) be a category. An invariant \( \Phi \) of objects of \( \mathcal{E} \) is balanced if it ‘looks at objects from all directions’: formally, for every automorphism \( F \) of \( \mathcal{E} \),

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The (invertible) spectrum is a balanced invariant
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We have invariants $\text{Spec}$ and $\text{Spec}^\times$ on $\text{Endo}(\text{FDVect})$ (the category of operators).
The (invertible) spectrum is a balanced invariant

We have invariants Spec and Spec$^\times$ on Endo($\text{FDVect}$) (the category of operators).

This category has some obvious automorphisms: those of the form $T \mapsto \alpha T + \beta$, where $\alpha, \beta \in k$ with $\alpha \neq 0$.
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Fix $\alpha$ and $\beta$. For all operators $T$,

$$\text{Spec}(\alpha T + \beta) = \alpha \text{Spec}(T) + \beta$$
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Fix $\alpha$ and $\beta$. For all operators $T$,

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so $\text{Spec}(T)$ determines $\text{Spec}(\alpha T + \beta)$. 
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Fix $\alpha$ and $\beta$. For all operators $T$,

$$\text{Spec}(\alpha T + \beta) = \alpha \text{Spec}(T) + \beta,$$

so Spec($T$) determines Spec($\alpha T + \beta$).

There are also some non-obvious automorphisms of Endo(\textbf{FDVect})!

Even so, it’s a fact that Spec($T$) determines Spec($\Phi(T)$) for all automorphisms $\Phi$ of Endo(\textbf{FDVect}).
The (invertible) spectrum is a balanced invariant

We have invariants \( \text{Spec} \) and \( \text{Spec}^\times \) on \( \text{Endo}(\text{FDVect}) \) (the category of operators).

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Fix \( \alpha \) and \( \beta \). For all operators \( T \),

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The same is true of \( \text{Spec}^\times \) (for slightly more subtle reasons).
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The same is true of \( \text{Spec}^\times \) (for slightly more subtle reasons).

In other words, \( \text{Spec} \) and \( \text{Spec}^\times \) are balanced invariants of linear operators.
5. The theorem
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**Theorem**

$\text{Spec}^x$ is the universal cyclic, balanced invariant of linear operators on finite-dimensional $k$-vector spaces.
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\( \text{Spec}^x \) is the universal cyclic, balanced invariant of linear operators on finite-dimensional \( k \)-vector spaces.

That is: let \( \Omega \) be a set and take \( \Phi: \text{ob} \left( \text{Endo}(\text{FDVect}) \right)/\cong \rightarrow \Omega \)
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$\text{Spec}^\times$ is the universal cyclic, balanced invariant of linear operators on finite-dimensional $k$-vector spaces.

That is: let $\Omega$ be a set and take $\Phi : \text{ob}(\text{Endo}(\text{FDVect}))/\cong \rightarrow \Omega$ such that

1. $\Phi(g \circ f) = \Phi(f \circ g)$ whenever $X \xleftarrow{g} \xrightarrow{f} Y$ in $\text{FDVect}$;
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for all linear operators \( T_1, T_2 \) and automorphisms \( F \) of \( \text{Endo}(\text{FDVect}) \).

Then there exists a unique \( \Phi \) such that

\[ \{\text{operators}\} \xrightarrow{\text{Spec}^\times} \{\text{finite subsets-with-multiplicity of } k^\times\} \]

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\begin{enumerate}
  \item \( \Phi(g \circ f) = \Phi(f \circ g) \) whenever \( X \xrightarrow{f} Y \) in \( \text{FDVect} \);
  \item \( \Phi(T_1) = \Phi(T_2) \implies \Phi(F(T_1)) = \Phi(F(T_2)) \)
    for all linear operators \( T_1, T_2 \) and automorphisms \( F \) of \( \text{Endo}(\text{FDVect}) \).
\end{enumerate}

Then there exists a unique \( \Phi \) such that

\[
\begin{array}{ccc}
\{ \text{operators} \} & \xrightarrow{\text{Spec}^\times} & \{ \text{finite subsets-with-multiplicity of } k^\times \} \\
\Phi & \downarrow & \Phi \\
& \Omega & \\
\end{array}
\]

commutes. (Thus, any such \( \Phi \) is a specialization of \( \text{Spec}^\times \).)
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**E.g.:** \( \text{tr} \) is cyclic and balanced
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Theorem

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That is: let $\Omega$ be a set and take $\Phi: \text{ob}(\text{Endo}(\text{FDVect}))/\cong \rightarrow \Omega$ such that

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for all linear operators $T_1, T_2$ and automorphisms $F$ of $\text{Endo}(\text{FDVect})$.

Then there exists a unique $\Phi$ such that

$$\xymatrix{
\{\text{operators}\} \ar[r]^{\text{Spec}^\times} \ar[d]_{\Phi} & \{\text{finite subsets-with-multiplicity of } k^\times\} \ar[l]_{\Phi} \\
\Omega }
$$

commutes. (Thus, any such $\Phi$ is a specialization of $\text{Spec}^\times$.)

E.g.: $\text{tr}$ is cyclic and balanced, and indeed $\text{tr}(T) = \sum_{\lambda \in k^\times} \alpha_T(\lambda) \cdot \lambda$. 

Imitating the (invertible) spectrum in other categories

What happens if you replace \( \text{FDVect} \) by a different category? That is: what if we look for the universal cyclic, balanced invariant of endomorphisms in \( C \), for some other category \( C \)?

Example: In \( \text{FinSet} \), a typical endomorphism looks like this: The universal cyclic, balanced invariant of endomorphisms in \( \text{FinSet} \) is \( X/T (\text{number of 1-cycles}, \text{number of 2-cycles}, \text{number of 3-cycles}, \ldots) \). That's the 'invertible spectrum' of an operator on a finite set.
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![Diagram of a typical endomorphism in FinSet](image-url)
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Postscript: Commutative rings and topos theory
From commutative rings to linear operators

The spectrum $\text{Spec} (R)$ of a (commutative) ring $R$ is the set of prime ideals of $R$, equipped with:

- a certain topology
- a certain sheaf of local rings.

The spectrum of a linear operator is a special case: Given an operator $T$, put $R(T) = k[x] / (\chi_T(x))$. Then the prime ideals of $R(T)$ are $(x - \lambda_1), \ldots, (x - \lambda_m)$ where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $T$.

Moreover, the stalks of the sheaf of local rings have dimensions $\alpha_T(\lambda_1), \ldots, \alpha_T(\lambda_m)$.

Thus, the linear-algebraic spectrum $\text{Spec} (T)$ can be recovered from the ring-theoretic spectrum $\text{Spec} (R(T))$.

But how can we understand the ring-theoretic spectrum abstractly?
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But how can we understand the ring-theoretic spectrum abstractly?
Hakim’s theorem

Monique Hakim (1986)

Book (1972)
Rings vs. local rings

Fact: The inclusion functor \( \text{local rings} \to \text{rings} \) has no adjoint.

Idea: Overcome this by allowing the ambient topos to vary.

Let \( \text{RingTopos} \) be the category of pairs \((E, R)\) where \( E \) is a topos and \( R \) is a ring in \( E \).

Let \( \text{LocRingTopos} \) be the category of pairs \((E, R)\) where \( E \) is a topos and \( R \) is a local ring in \( E \).

Fact: The inclusion functor \( \text{LocRingTopos} \to \text{RingTopos} \) has a right adjoint (for completely general reasons).

You can think of the adjoint as constructing the 'free local ring' on a ring: but it might live in a different topos from the ring you started with.
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The spectrum as the free local ring

Let $R$ be a ring. Then $R$ is a ring in $\mathbf{Set}$, so determines an object $(\mathbf{Set}, R)$ of $\mathbf{RingTopos}$. Also $\text{Spec}(R)$ is a topological space, giving a topos $\mathbf{Sh}(\text{Spec}(R))$. It comes with a sheaf of local rings, giving a local ring $\mathcal{O}_R$ in the topos $\mathbf{Sh}(\text{Spec}(R))$. So, the spectrum of $R$ determines an object of $\mathbf{LocRingTopos}$. Theorem (Hakim) The right adjoint to the inclusion $\mathbf{LocRingTopos}$ maps $(\mathbf{Set}, R)$ to $(\mathbf{Sh}(\text{Spec}(R)), \mathcal{O}_R)$, for all rings $R$. In this sense, $\text{Spec}$ is exactly that right adjoint.
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It comes with a sheaf of local rings, giving a local ring \( O_R \) in the topos \( \textbf{Sh}(\text{Spec}(R)) \).
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So, the spectrum of $R$ determines an object $(\textbf{Sh}(\text{Spec}(R)), O_R)$ of $\textbf{LocRingTopos}$. 

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Theorem (Hakim)

*The right adjoint to the inclusion $\text{LocRingTopos} \hookrightarrow \text{RingTopos}$ maps $(\textbf{Set}, R)$ to $(\text{Sh}(\text{Spec}(R)), O_R)$, for all rings $R$.*
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Perspectives

Hakim's theorem describes a universal property of the spectrum of a ring. The spectrum of a linear operator is a special case of the spectrum of a ring. So, this gives an abstract characterization of the spectrum of an operator.

However:

- To make the step from operators to rings, we used the characteristic polynomial. What is its place abstractly?
- The characterization of the spectrum of an operator coming from Hakim's theorem is less direct than the one established in this talk, which stays within the topos of sets.
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