

# Interacting Frobenius Algebras

Ross Duncan & Kevin Dunne

University of Strathclyde

June 2016

# Symmetric Monoidal Categories

## Definition

A strict symmetric monoidal category  $(\mathcal{C}, \otimes, I)$  consists of

- ▶ Objects  $A, B, C, \dots$
- ▶ Morphisms  $f : A \rightarrow B$
- ▶ Monoidal product  $\otimes$

$$f : A \rightarrow B \qquad g : C \rightarrow D$$

$$f \otimes g : A \otimes B \rightarrow C \otimes D$$

# Symmetric Monoidal Categories

A *dagger* on  $(\mathcal{C}, \otimes, I)$  consists of an involutive symmetric monoidal functor

$$\dagger : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$$

i.e. every morphism has an adjoint

$$A \xrightarrow{f} B \qquad B \xrightarrow{f^\dagger} A$$

$$f^{\dagger\dagger} = f$$

## Definition

An isomorphism  $f : A \rightarrow B$  is called *unitary* if  $f^{-1} = f^\dagger$ .

# Frobenius Algebras

## Definition

A  $\dagger$ -special commutative Frobenius algebra ( $\dagger$ -SCFA) in  $(\mathcal{C}, \otimes, I)$  consists of: An object  $A \in \mathcal{C}$ ,

$$\mu : A \otimes A \rightarrow A, \quad \eta : I \rightarrow A$$

$$\mu^\dagger : A \rightarrow A \otimes A, \quad \eta^\dagger : A \rightarrow I$$

satisfying...

# Frobenius Algebras

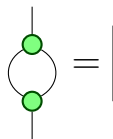
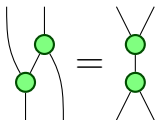
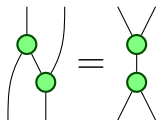
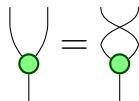
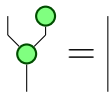
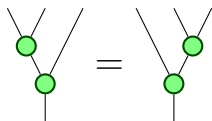
## Definition

A  $\dagger$ -special commutative Frobenius algebra ( $\dagger$ -SCFA) in  $(\mathcal{C}, \otimes, I)$  consists of: An object  $A \in \mathcal{C}$ ,

$$\begin{aligned}\mu &= \text{green circle with two lines entering from top-left and top-right, one line exiting from bottom}, & \eta &= \text{green circle with one line entering from bottom} \\ \mu^\dagger &= \text{green circle with one line entering from top, two lines exiting from bottom-left and bottom-right}, & \eta^\dagger &= \text{green circle with one line entering from top}\end{aligned}$$

satisfying...

# Frobenius Algebras



## Observables are Frobenius Algebras

Let  $\{ |e_i\rangle \}_{i \in I}$  be an orthonormal basis. Define the  $\dagger$ -SCFA:

$$\begin{aligned} H &\xrightarrow{\mu^\dagger} H \otimes H \\ |e_i\rangle &\longmapsto |e_i\rangle \otimes |e_i\rangle \end{aligned}$$

# Observables are Frobenius Algebras

Let  $\{ |e_i\rangle \}_{i \in I}$  be an orthonormal basis. Define the  $\dagger$ -SCFA:

$$\begin{aligned} H &\xrightarrow{\mu^\dagger} H \otimes H \\ |e_i\rangle &\longmapsto |e_i\rangle \otimes |e_i\rangle \end{aligned}$$

## A new description of orthogonal bases

Bob Coecke, Dusko Pavlovic and Jamie Vicary  
Oxford University Computing Laboratory

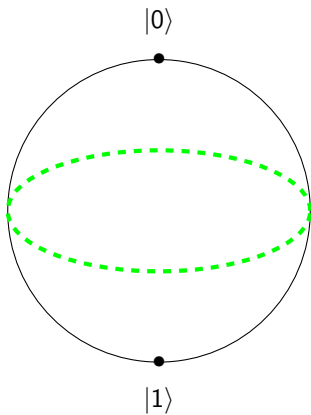
Theorem (Coecke, Pavlovic, Vicary)

Every  $\dagger$ -SCFA in **fdHilb** is of this form.



## Observables are Frobenius Algebras

*“Hence orthogonal and orthonormal bases can be axiomatised in terms of composition of operations and tensor product only, without any explicit reference to the underlying vector spaces.”*



$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = Z_\alpha : Q \rightarrow Q$$

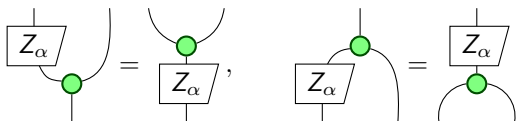
# Observables and Phases

A basis  $\{ |0\rangle, |1\rangle \}$

$$Z_\alpha |0\rangle = |0\rangle$$

$$Z_\alpha |1\rangle = -|1\rangle$$

A  $\dagger$ -SCFA  $\{ \text{green circle with two lines} \}$



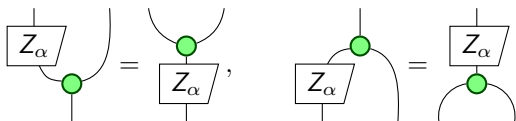
# Observables and Phases

A basis  $\{ |0\rangle, |1\rangle \}$

$$Z_\alpha |0\rangle = |0\rangle$$

$$Z_\alpha |1\rangle = |1\rangle$$

A  $\dagger$ -SCFA  $\{ \text{green circle with legs} \}$



Call  $Z_\alpha$  the *phases* for this Frobenius algebra

or, the  $\bullet$ -*phases*

# Phase Groups and Unbiased Points

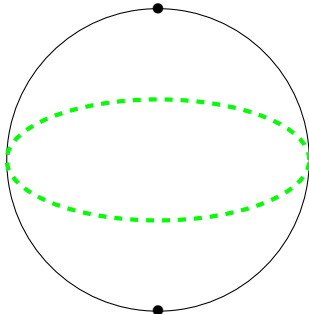
$\hat{g}$  is called  $\bullet$ -unbiased if it is of the form

$$\hat{g} = \begin{array}{c} \bullet \\ \triangle \hat{g} \\ | \end{array} = \begin{array}{c} \bullet \\ \square g \\ | \end{array} \quad \text{for a } \bullet\text{-phase } g.$$

The  $\bullet$ -unbiased points are isomorphic to the  $\bullet$ -phase group.

$$\begin{array}{c} \bullet \quad \bullet \\ \square g \quad \square h \\ \quad \quad \quad \bullet \\ \quad \quad \quad | \end{array} \cong \begin{array}{c} \square g \\ \square h \\ | \end{array}$$

$|0\rangle$



$|1\rangle$

# Algebraic Theories - PROPs

## Definition

- ▶ A PROP is a strict symmetric monoidal category whose objects are generated by a single object via the tensor product.
- ▶ A PROP is a strict symmetric monoidal category with objects the natural numbers.

# Algebraic Theories - PROPs

## Definition

- ▶ A PROP is a strict symmetric monoidal category whose objects are generated by a single object via the tensor product.
- ▶ A PROP is a strict symmetric monoidal category with objects the natural numbers.

## Definition

A  $\dagger$ -PROP is a PROP with a dagger.



# Algebraic Theories - PROPs

Algebras of PROPs

$$F : \mathbf{A} \rightarrow \mathcal{C}$$

## Example

$$\mathbf{M} = (\Sigma, E)$$

$$\Sigma = \{ \text{⋈}, \text{⊖} \}$$

$$E = \left\{ \begin{array}{l} \text{⋈} = \text{⋈} , \quad \text{⊖} = \text{⊖} , \quad \text{⊖} = \text{⊖} \end{array} \right\}$$

*“ $\mathbf{M}$  is the free theory of commutative monoids”*

## Example

$$\mathbf{M}^{op} = (\Sigma, E)$$

$$\Sigma = \{ \text{green circle with two lines}, \text{green circle with one line} \}$$

$$E = \left\{ \begin{array}{l} \text{green circle with two lines} = \text{green circle with two lines} \\ \text{green circle with one line} = \text{green circle with one line} \\ \text{green circle with two lines} = \text{green circle with one line} \\ \text{green circle with one line} = \text{green circle with one line} \end{array} \right\}$$

*" $\mathbf{M}^{op}$  is the free theory cocommutative comonoids"*

## New PROPs From Old

Quotients of PROPs:  $\mathbf{T} = (\Sigma, E)$

$$\mathbf{T}/E' := (\Sigma, E \sqcup E')$$

## New PROPs From Old

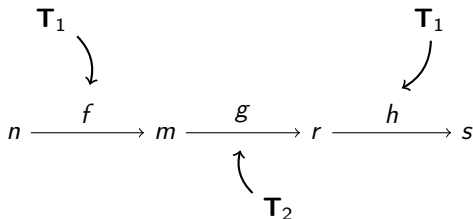
Quotients of PROPs:  $\mathbf{T} = (\Sigma, E)$

$$\mathbf{T}/E' := (\Sigma, E \sqcup E')$$

Coproduct of PROPs:  $\mathbf{T}_1 = (\Sigma_1, E_1)$  and  $\mathbf{T}_2 = (\Sigma_2, E_2)$

$$\mathbf{T}_1 + \mathbf{T}_2 := (\Sigma_1 \sqcup \Sigma_2, E_1 \sqcup E_2)$$

Expressions in  $\mathbf{T}_1 + \mathbf{T}_2$ :



# Composing PROPs

*Theory and Applications of Categories*, Vol. 13, No. 9, 2004, pp. 147–163.

## COMPOSING PROPS

*Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday*

STEPHEN LACK

“ $\mathbf{T}_2$  composed with  $\mathbf{T}_1$ ”

$\mathbf{T}_2; \mathbf{T}_1$

$(\mathbf{T}_1 + \mathbf{T}_2)/E \stackrel{?}{=} \mathbf{T}_2; \mathbf{T}_1$

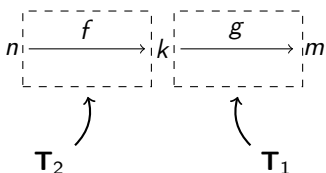
## Composing PROPs

Q: when is  $(\mathbf{T}_1 + \mathbf{T}_2)/E$  a composition  $\mathbf{T}_2; \mathbf{T}_1$ ?

## Composing PROPs

Q: when is  $(\mathbf{T}_1 + \mathbf{T}_2)/E$  a composition  $\mathbf{T}_2; \mathbf{T}_1$ ?

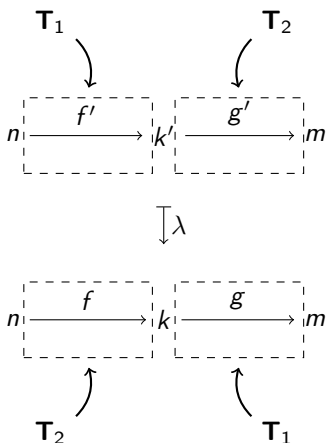
A: when every morphism  $h : n \rightarrow m$  is of the form





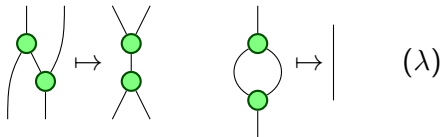
# Composing PROPs

This amounts to giving rewrite rules:



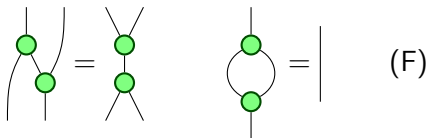
# Examples

$$\mathbf{M} + \mathbf{M}^{\text{op}}$$



# Examples

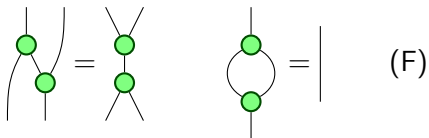
$$\mathbf{M} + \mathbf{M}^{\text{op}}$$



$$\mathbf{F} := (\mathbf{M} + \mathbf{M}^{\text{op}}) / \mathbf{F} = \mathbf{M}; \mathbf{M}^{\text{op}}$$

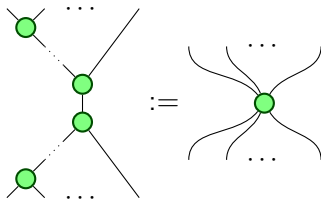
# Examples

$$\mathbf{M} + \mathbf{M}^{\text{op}}$$



$$\mathbf{F} := (\mathbf{M} + \mathbf{M}^{\text{op}}) / \mathbf{F} = \mathbf{M}; \mathbf{M}^{\text{op}}$$

*“F is the free theory of Spiders”*

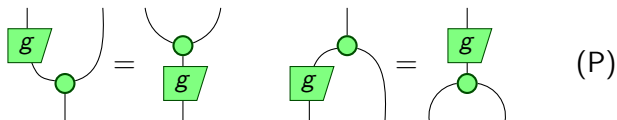


## Examples

Let  $G$  be an abelian group. Define the PROP  $\mathbf{G}$

$$\Sigma = \{g : 1 \rightarrow 1 \mid g \in G\}, \quad E = \{g \circ h = gh\}$$

Consider  $\mathbf{F} + \mathbf{G}$  and equations:

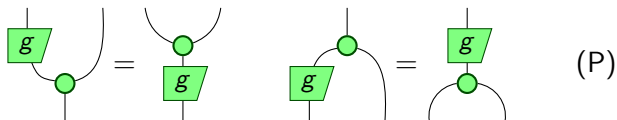


## Examples

Let  $G$  be an abelian group. Define the PROP  $\mathbf{G}$

$$\Sigma = \{g : 1 \rightarrow 1 \mid g \in G\}, \quad E = \{g \circ h = gh\}$$

Consider  $\mathbf{F} + \mathbf{G}$  and equations:



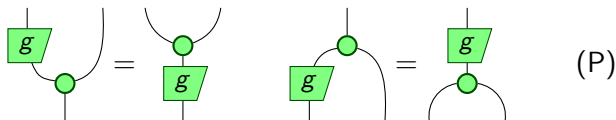
$$\mathbf{FG} := (\mathbf{F} + \mathbf{G})/P$$

## Examples

Let  $G$  be an abelian group. Define the PROP  $\mathbf{G}$

$$\Sigma = \{g : 1 \rightarrow 1 \mid g \in G\}, \quad E = \{g \circ h = gh\}$$

Consider  $\mathbf{F} + \mathbf{G}$  and equations:



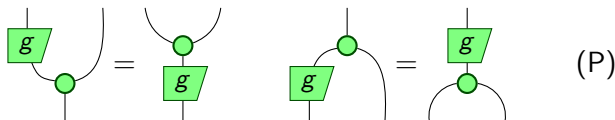
$$\mathbf{FG} := (\mathbf{F} + \mathbf{G})/P = \mathbf{M}; \mathbf{G}; \mathbf{M}^{\text{op}}$$

## Examples

Let  $G$  be an abelian group. Define the PROP  $\mathbf{G}$

$$\Sigma = \{g : 1 \rightarrow 1 \mid g \in G\}, \quad E = \{g \circ h = gh\}$$

Consider  $\mathbf{F} + \mathbf{G}$  and equations:

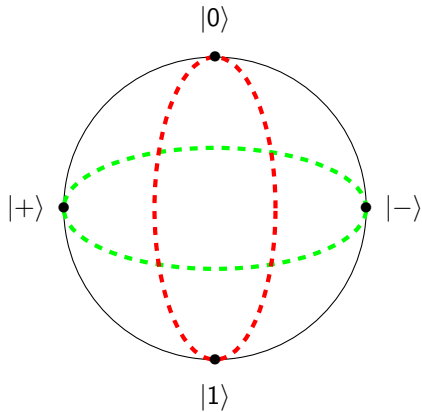


$$\mathbf{FG} := (\mathbf{F} + \mathbf{G})/P = \mathbf{M}; \mathbf{G}; \mathbf{M}^{\text{op}}$$

*“FG is the free theory of an observable with phase group  $G$ ”*

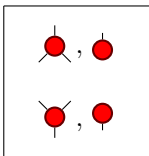
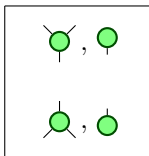
*“FG is the free theory phased Spiders”*





**Frobenius**

**Frobenius**



# Strongly Complementary Observables

The (scaled) bialgebra equations

The image shows three equations involving colored nodes and lines. The first equation shows a red node with two incoming lines on top and one outgoing line on the bottom, connected to a green node with two outgoing lines on top and one incoming line on the bottom. This is equal to a diagram where two green nodes are connected to two red nodes in a crossing pattern, with an additional red node and a green node to the right. The second equation shows two red nodes with one incoming line on the bottom, equal to two red nodes with one outgoing line on the top. The third equation shows two red nodes with one incoming line on the top, equal to two green nodes with one outgoing line on the bottom. The label (B) is on the right.

$$\begin{array}{c} \text{Red node with 2 incoming lines on top, 1 outgoing line on bottom} \\ \text{Green node with 2 outgoing lines on top, 1 incoming line on bottom} \end{array} = \begin{array}{c} \text{Green node with 2 outgoing lines on top, 1 incoming line on bottom} \\ \text{Red node with 2 outgoing lines on top, 1 incoming line on bottom} \\ \text{Red node with 2 outgoing lines on top, 1 incoming line on bottom} \\ \text{Green node with 2 outgoing lines on top, 1 incoming line on bottom} \end{array}, \quad \begin{array}{c} \text{Red node with 1 incoming line on bottom} \\ \text{Red node with 1 incoming line on bottom} \end{array} = \begin{array}{c} \text{Red node with 1 outgoing line on top} \\ \text{Red node with 1 outgoing line on top} \end{array}, \quad \begin{array}{c} \text{Red node with 1 incoming line on top} \\ \text{Red node with 1 incoming line on top} \end{array} = \begin{array}{c} \text{Green node with 1 outgoing line on bottom} \\ \text{Green node with 1 outgoing line on bottom} \end{array} \quad (\text{B})$$

# Strongly Complementary Observables


$$\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \circlearrowleft \quad \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} = \circlearrowright \quad (*)$$

## Theorem

*The morphism*

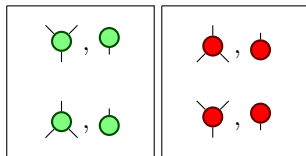


*is an antipode for both bialgebras iff the equations (\*) hold.*

# Strongly Complementary Observables

**Frobenius**

**Frobenius**



**Hopf**



**Hopf**



## Strongly Complementary Observables

$$\mathbf{IF}(G, H) := (\mathbf{FG} + \mathbf{FH})/B^*$$

## Strongly Complementary Observables

$$\mathbf{IF}(G, H) := (\mathbf{FG} + \mathbf{FH})/B^*$$

*“ $\mathbf{IF}(G, H)$  is the free theory of a pair of strongly complementary observables with given phase groups”*

# Interacting Hopf Algebras

Filippo Bonchi<sup>a</sup>, Paweł Sobociński<sup>b</sup>, Fabio Zanasi<sup>c</sup>

<sup>a</sup>*ENS de Lyon, Université de Lyon, CNRS, INRIA, France*

<sup>b</sup>*University of Southampton, United Kingdom*

<sup>c</sup>*Radboud University of Nijmegen, Netherlands*

## Strongly Complementary Observables

Q: Is  $\mathbf{IF}(G, H)$  a composition  $\mathbf{FG}; \mathbf{FH}$ ?



## Strongly Complementary Observables

Q: Is  $\mathbf{IF}(G, H)$  a composition  $\mathbf{FG}; \mathbf{FH}$ ?

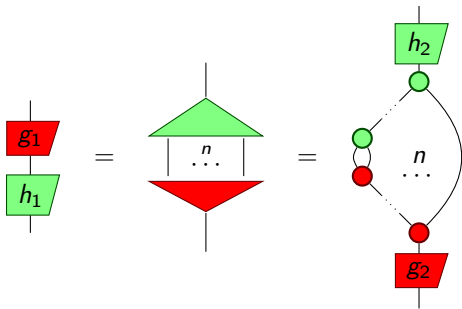
Theorem

*No.*

## Theorem

The PROP  $\mathbf{IF}(G, H)$  is not a composition  $\mathbf{FG}; \mathbf{FH}$ .

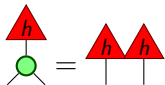
If it were a composition...



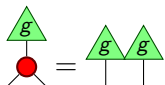
# Set-Like Elements

## Definition

A morphism  $h : 0 \rightarrow 1$  is called  $\bullet$ -set-like if



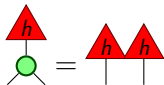
A morphism  $g : 0 \rightarrow 1$  is called  $\bullet$ -set-like if



# Set-Like Elements

## Definition

A morphism  $h : 0 \rightarrow 1$  is called  $\bullet$ -set-like if



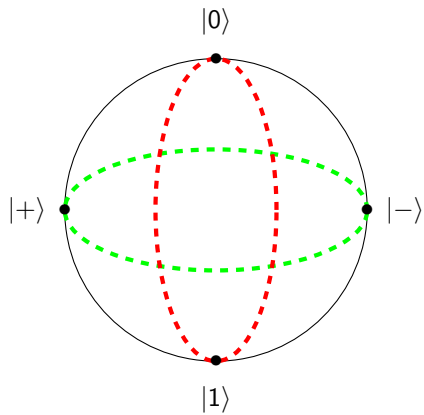
A morphism  $g : 0 \rightarrow 1$  is called  $\bullet$ -set-like if



## Definition

Let  $K_{\bullet}$  be the collection of  $\bullet$ -set-like elements. We say there are *enough  $\bullet$ -set-like elements* if for any  $f, f' : 1 \rightarrow 1$ :

$$\forall g \in K_{\bullet}, \quad f \circ g = f' \circ g \quad \Rightarrow \quad f = f'$$






# Set-Like Elements are Unbiased

## Lemma

*The -set-like elements are a subgroup of the -unbiased points.*

# Interacting Observables With Set-Like Elements

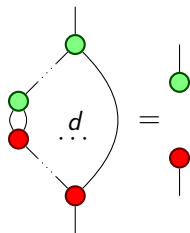
$$\mathbf{IFK}_d(G \geq G_K, H \geq H_K)$$

- ▶ -set-like elements  $H_K$
- ▶ -set-like elements  $G_K$
- ▶ enough -set-like elements.

# Proving the Theorem

## Lemma

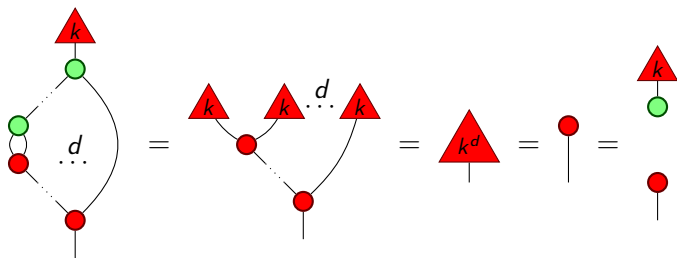
If  $H_K$  is a finite group with exponent  $d$ , then





## Proving the Theorem

*Proof.* For  $k \in K_{\bullet}$



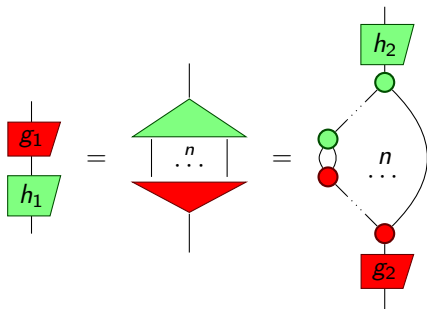
Since we have enough  $\bullet$ -set-like elements, we are done.

# Proving the Theorem

## Theorem

The PROP  $IF(G, H)$  is not a composition  $FG; FH$ .

*Proof.*

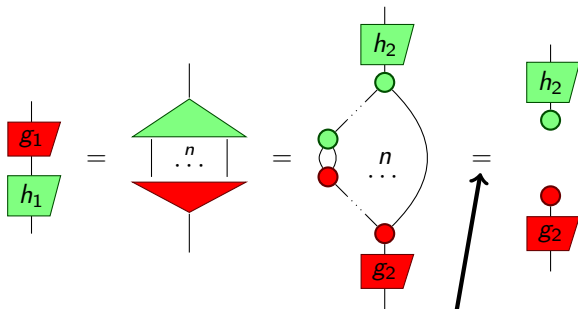


# Proving the Theorem

## Theorem

The PROP  $IF(G, H)$  is not a composition  $FG; FH$ .

Proof.



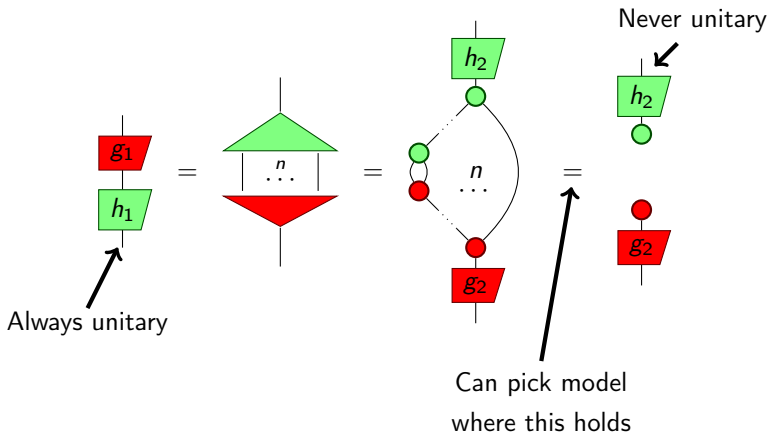
Can pick model  
where this holds

# Proving the Theorem

## Theorem

The PROP  $IF(G, H)$  is not a composition  $FG; FH$ .

Proof.



## Conclusion

- ▶ There is no hope.

## Conclusion

- ▶ There is no hope.
- ▶ Well OK, there might be some hope...
- ▶ Generalised Euler decomposition

## Conclusion

- ▶ There is no hope.
- ▶ Well OK, there might be some hope...
- ▶ Generalised Euler decomposition
- ▶ Recover other aspects of ZX calculus, e.g. the Haddamard