Interacting Frobenius Algebras

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Symmetric Monoidal Categories

Definition

A strict symmetric monoidal category $(\mathcal{C}, \otimes, I)$ consists of

- Objects $A, B, C, \ldots$
- Morphisms $f : A \to B$
- Monoidal product $\otimes$

$$f : A \to B \qquad\qquad g : C \to D$$

$$f \otimes g : A \otimes B \to C \otimes D$$

Symmetric Monoidal Categories

A *dagger* on $(\mathcal{C}, \otimes, I)$ consists of an involutive symmetric monoidal functor

$$\dagger : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$$

i.e. every morphism has an adjoint

$$A \xrightarrow{\quad f \quad} B \qquad\qquad B \xrightarrow{\quad f^\dagger \quad} A$$

$$f^{\dagger\dagger} = f$$

Definition

An isomorphism $f : A \to B$ is called *unitary* if $f^{-1} = f^\dagger$.

Frobenius Algebras

Definition

A †-*special commutative Frobenius algebra* (†-SCFA) in $(\mathcal{C}, \otimes, I)$ consists of: An object $A \in \mathcal{C}$,

$$\mu : A \otimes A \to A, \qquad \eta : I \to A$$

$$\mu^\dagger : A \to A \otimes A, \qquad \eta^\dagger : A \to I$$

satisfying...

Frobenius Algebras

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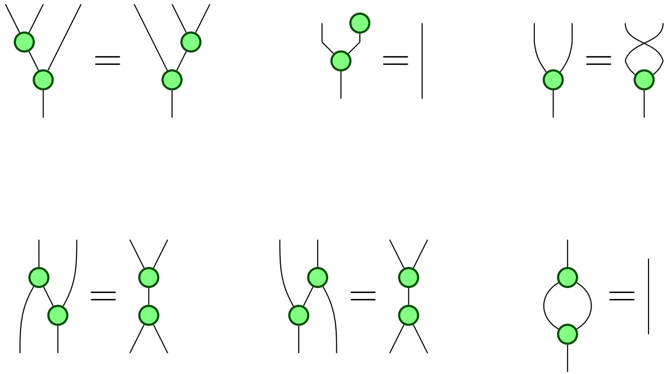
Frobenius Algebras



Observables are Frobenius Algebras

Let $\{\,|e_i\rangle\,\}_{i\in I}$ be an orthonormal basis. Define the †-SCFA:

$$\begin{aligned} H &\xrightarrow{\mu^\dagger} H \otimes H \\ |e_i\rangle &\longmapsto |e_i\rangle \otimes |e_i\rangle \end{aligned}$$

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A new description of orthogonal bases

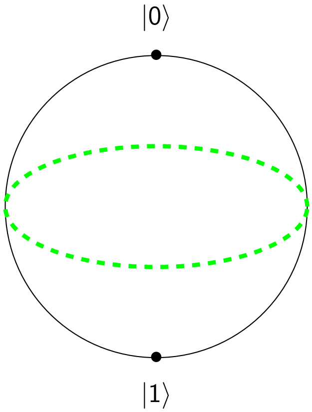
Bob Coecke, Dusko Pavlovic and Jamie Vicary
Oxford University Computing Laboratory

Theorem (Coecke, Pavlovic, Vicary)

*Every †-SCFA in **fdHilb** is of this form.*

Observables are Frobenius Algebras

"Hence orthogonal and orthonormal bases can be axiomatised in terms of composition of operations and tensor product only, without any explicit reference to the underlying vector spaces."



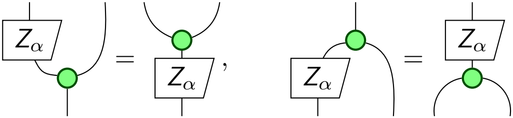
$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = Z_\alpha : Q \to Q$$

Observables and Phases

A basis $\{ \, |0\rangle \, , \; |1\rangle \, \}$

$$Z_\alpha |0\rangle = |0\rangle$$
$$Z_\alpha |1\rangle = |1\rangle$$

A †-SCFA {  }



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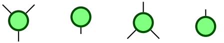
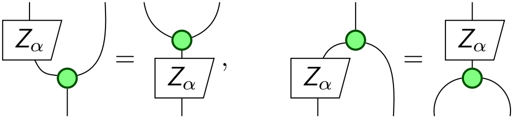
A †-SCFA {  }

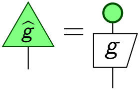


Call Z_α the *phases* for this Frobenius algebra

or, the -*phases*

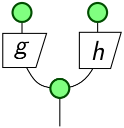
Phase Groups and Unbiased Points

$\widehat{g}$ is called ○-*unbiased* if it is of the form

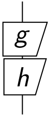


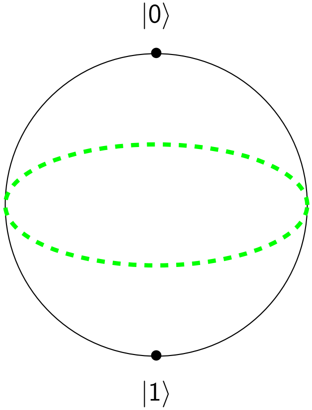
for a ○-phase g.

The ○-unbiased points are isomorphic to the ○-phase group.



$\cong$




$|0\rangle$
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Algebraic Theories - PROPs

Definition

- A PROP is a strict symmetric monoidal category whose objects are generated by a single object via the tensor product.
- A PROP is a strict symmetric monoidal category with objects the natural numbers.

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Definition

A †-PROP is a PROP with a dagger.

Algebraic Theories - PROPs

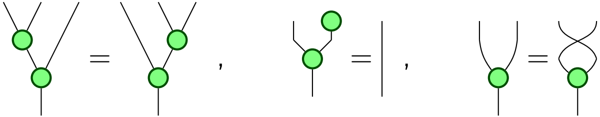
Algebras of PROPs

$$F : \mathbf{A} \to \mathcal{C}$$

Example

$$\mathbf{M} = (\Sigma, E)$$

$\Sigma = \{$  $\}$

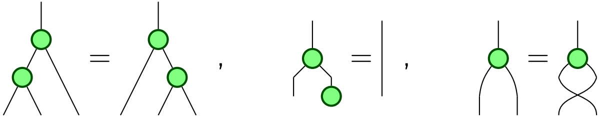
$E = \{$  $\}$

*"**M** is the free theory of commutative monoids"*

Example

$$\mathbf{M}^{op} = (\Sigma, E)$$

$\Sigma = \{$  ,  $\}$

$E = \{$  $\}$

"M^{op} is the free theory cocommutative comonoids"

New PROPs From Old

Quotients of PROPs: $\mathbf{T} = (\Sigma, E)$

$$\mathbf{T}/E' := (\Sigma, E \sqcup E')$$

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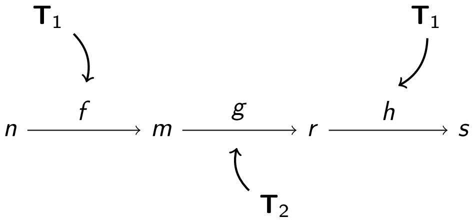
Coproduct of PROPs: $\mathbf{T}_1 = (\Sigma_1, E_1)$ and $\mathbf{T}_2 = (\Sigma_2, E_2)$

$$\mathbf{T}_1 + \mathbf{T}_2 := (\Sigma_1 \sqcup \Sigma_2, E_1 \sqcup E_2)$$

Expressions in $\mathbf{T}_1 + \mathbf{T}_2$:



Composing PROPs

Theory and Applications of Categories, Vol. 13, No. 9, 2004, pp. 147–163.

COMPOSING PROPS

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday

STEPHEN LACK

"$\mathbf{T}_2$ composed with $\mathbf{T}_1$"

$$\mathbf{T}_2 ; \mathbf{T}_1$$

$$(\mathbf{T}_1 + \mathbf{T}_2)/E \stackrel{?}{=} \mathbf{T}_2 ; \mathbf{T}_1$$

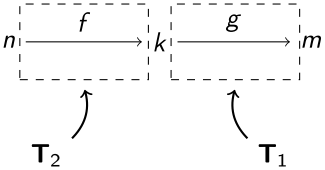
Composing PROPs

Q: when is $(\mathbf{T}_1 + \mathbf{T}_2)/E$ a composition $\mathbf{T}_2 ; \mathbf{T}_1$?

Composing PROPs

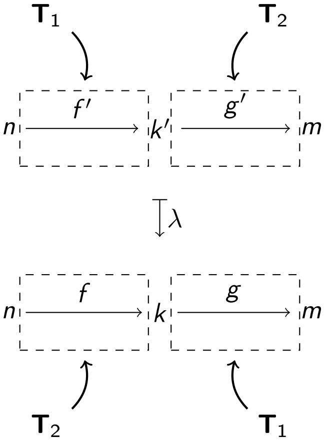
Q: when is $(\mathbf{T}_1 + \mathbf{T}_2)/E$ a composition $\mathbf{T}_2 ; \mathbf{T}_1$?

A: when every morphism $h : n \to m$ is of the form



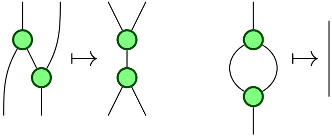
Composing PROPs

This amounts to giving rewrite rules:



Examples

$$\mathsf{M} + \mathsf{M}^{\mathsf{op}}$$



(λ)

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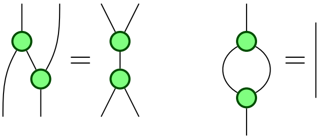

(F)

$$\mathbf{F} := (\mathbf{M} + \mathbf{M}^{\mathsf{op}})/F = \mathbf{M}; \mathbf{M}^{\mathsf{op}}$$

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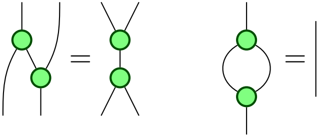

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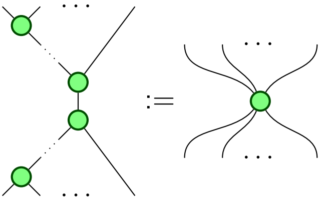
"***F*** *is the free theory of Spiders*"

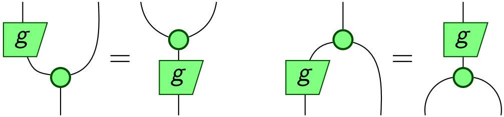


Examples

Let G be an abelian group. Define the PROP $\mathbf{G}$

$$\Sigma = \{g : 1 \to 1 \mid g \in G\}, \qquad E = \{g \circ h = gh\}$$

Consider $\mathbf{F} + \mathbf{G}$ and equations:



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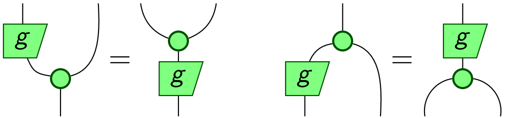

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$$\mathbf{F}G \;:=\; (\mathbf{F} + \mathbf{G})/P$$

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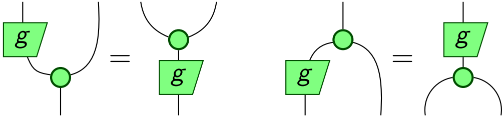

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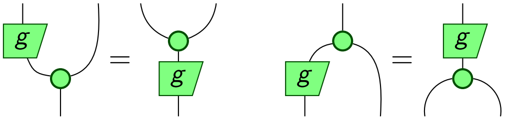
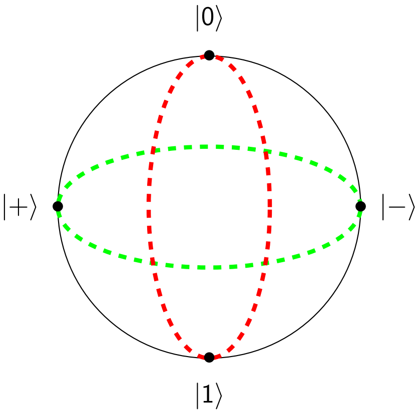


(P)

$$\mathbf{F}G := (\mathbf{F} + \mathbf{G})/P = \mathbf{M}; \mathbf{G}; \mathbf{M}^{\mathrm{op}}$$

*"**F**G is the free theory of an observable with phase group G"*

*"**F**G is the free theory phased Spiders"*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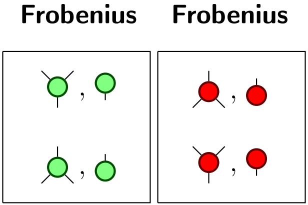
Frobenius
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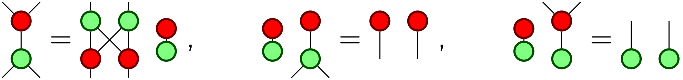
Strongly Complementary Observables

The (scaled) bialgebra equations



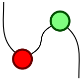
(B)

Strongly Complementary Observables

(∗)

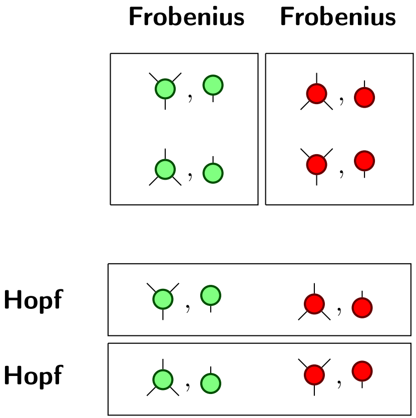
Theorem

The morphism



is an antipode for both bialgebras iff the equations (∗) hold.

Strongly Complementary Observables



Strongly Complementary Observables

$$\mathbf{IF}(G, H) := (\mathbf{F}G + \mathbf{F}H)/B*$$

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$$\mathbf{IF}(G, H) := (\mathbf{F}G + \mathbf{F}H)/B*$$

*"**IF**(G, H) is the free theory of a pair of strongly complementary observables with given phase groups"*

Interacting Hopf Algebras

Filippo Bonchi[a], Paweł Sobociński[b], Fabio Zanasi[c]

[a] *ENS de Lyon, Université de Lyon, CNRS, INRIA, France*
[b] *University of Southampton, United Kingdom*
[c] *Radboud University of Nijmegen, Netherlands*

Strongly Complementary Observables

Q: Is $\mathbf{IF}(G, H)$ a composition $\mathbf{F}G; \mathbf{F}H$?

Strongly Complementary Observables

Q: Is $\mathbf{IF}(G, H)$ a composition $\mathbf{F}G; \mathbf{F}H$?

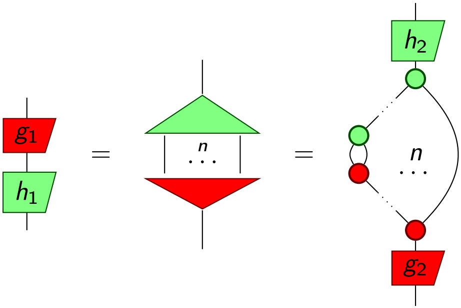
Theorem

No.

Theorem

The PROP $\mathbf{IF}(G, H)$ is not a composition $\mathbf{F}G ; \mathbf{F}H$.

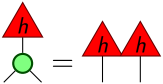
If it were a composition...



Set-Like Elements

Definition

A morphism $h : 0 \to 1$ is called -*set-like* if



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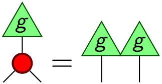
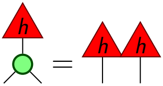


Set-Like Elements

Definition

A morphism $h : 0 \to 1$ is called ○-*set-like* if



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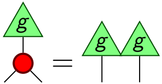
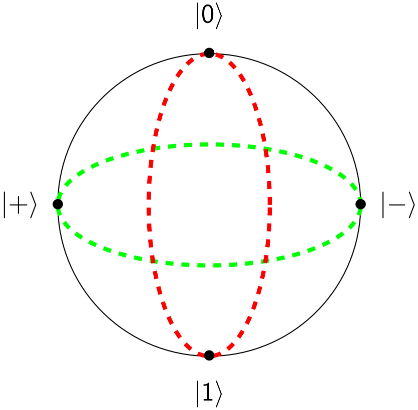


Definition

Let $K_{\circ}$ be the collection of ○-set-like elements. We say there are *enough* ○*-set-like elements* if for any $f, f' : 1 \to 1$:

$$\forall g \in K_{\circ}, \quad f \circ g = f' \circ g \quad \Rightarrow \quad f = f'$$


$|0\rangle$
$|+\rangle$
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Set-Like Elements are Unbiased

Lemma

The  *-set-like elements are a subgroup of the*  *-unbiased points.*

Interacting Observables With Set-Like Elements

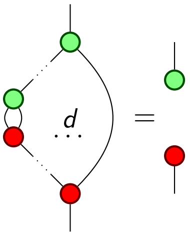
$$\mathbf{IFK_d}(G \geq G_K, H \geq H_K)$$

- ●-set-like elements H_K
- ●-set-like elements G_K
- enough ●-set-like elements.

Proving the Theorem

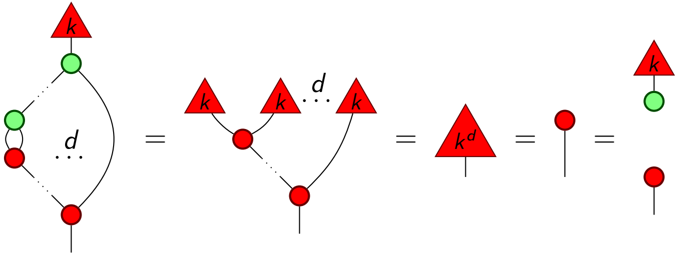
Lemma

If H_K is a finite group with exponent d, then



Proving the Theorem

Proof. For $k \in K_{\circ}$



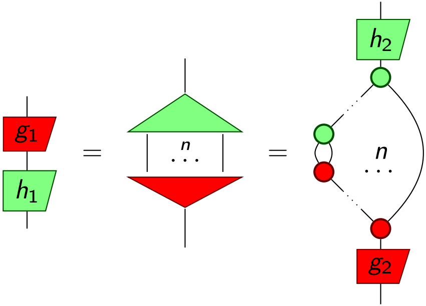
Since we have enough ○-set-like elements, we are done.

Proving the Theorem

Theorem

The PROP $\mathbf{IF}(G, H)$ *is not a composition* $\mathbf{F}G ; \mathbf{F}H$.

Proof.



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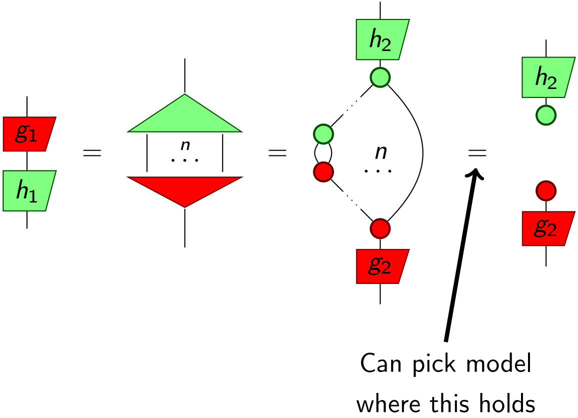

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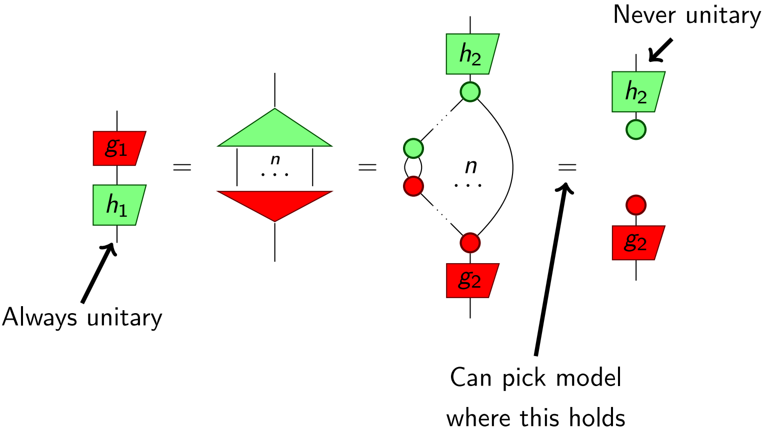

Conclusion

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- Generalised Euler decomposition
- Recover other aspects of ZX calculus, e.g. the Haddamard